# GENERAL EQUILIBRIUM WITH PUBLIC GOODS: A GEOMETRIC APPROACH

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**Abstract.**This paper demonstrates a certain symmetry between the private-private and public-private commodity spaces and then uses this symmetry in order to develop the geometry essential for the construction of an Edgeworth box when one of the commodities is a public good. The apparatus is next employed to point out certain flaws in previous attempts to generalize the Edgeworth box analysis, and to correct them. Finally it proves that it is impossible to make someone worse off by offering him a free public good when individuals are identified with distinct production blocks, thus negating an assertion made in the literature to the contrary.

## I. Introduction

The Edgeworth-Bowley box has been proven a simple yet powerful geometric technique in the analysis of two-by-two general equilibrium models. Its influence far exceeds that of a simple pedagogic device, and it has enhanced the understanding and insights of the working of general equilibrium models, especially in microeconomics and the pure theory of international trade.

Use of the Edgeworth diagram however, has thus far so as to include a public good been restricted in models with private goods. Fairly recent steps toward a useful generalization have been taken (cf. Samuelson, 1955; Dolbear, 1967; McGuire and Aaron, 1969; Shibata,1971; Ley, 1996), the closest to a complete adaptation being that of Shibata [1].

1. Johansen (1963) presents a model that resembles a general equilibrium model, but is actually a partial equilibrium model supplemented with (redefined) indifference curves.

In his article published in the *Journal of Political Economy*, Shibata demonstrates a specific adaptation concerning the use of the Edgeworth box, such as to include the case in which one of the two commodities is a public good. The resulting geometry is claimed to be adequate to support, among the things, a remarkable proposition - the possibility of immiserizing effects of a freely-provided public good. Shibata asserts (p.27) that the net welfare effects of such provision might be negative to the recipient, because the primary gain from the free public good might be offset by changes in factor earnings.

Such a counter-intuitive proposition and the analysis accompanying it merit careful consideration. After such consideration, it is argued that Shibata's geometric analysis is flawed. The way he constructs the Edgeworth box for a private and a public good has a deficiency which, although harmless when the analysis is limited to constant prices (by virtue of a geometric coincidence), becomes very serious indeed when the analusis is extended to include variable prices (increasing marginal costs).

This paper first provides the geometry essential for the sound construction of an Edgeworth box when one of the commodities is a public good. With the assistance of the geometry, a certain symmetry between the two uses of the Edgeworth technique is demonstrated. Second, the insights gained are used to trace the origin and consequences of the flaws in Shibata's analysis. Finally, it is shown that it is impossible to make someone worse off by offering him a free public good when individuals are indentified with distinct production blocks (i.e. with productive activity rather than a single factor of production), as is the case with Shibata's analysis.

## II. The Edgeworth Box on Private-Public Commodity Space

The Edgeworth box in its standard use describes a space that is the overlap of three coordinate systems, one for each of the actors and their sum - the community. The origins of these coordinate system are defined and oriented in respect to each other, so that the pair of coordinates of any given point X in the described overlap satisfy the following relations:

$$X_{1A} + X_{1B} = X_{1C} \text{ and } X_{2A} + X_{2B} = X_{2C}; \qquad (1)$$

$$l_A = l_B = l_C \qquad (2)$$

where A, B and C are the two actors and the community, respectively, and $X_{ik}$ the quantity of the $i^{th}$ commodity measured from origin k. Furthermore, the slope $l$ of any given line in the described space as measured from each of the three different origins is such that it complies with (2). The community origin C can coincide with either origin A or B without any effect on the described relationships. With $X_{ik}$ describing the name and quantity of the commodity that goes to the k[th] actor, and $l_k$ describing prices and marginal rates of substitution for k, the standard Edgeworth box can be defined for any given (fixed) amounts of $X_1$ and $X_2$.

When one of the commodities is a public good, however, the overlap of the three coordinate systems must be defined so as to satisfy a different set of conditions, dictated by the nature of the public good. Thus, we must have:

For any point X in the overlap,

$$X_{1A} + X_{1B} = X_{1C}\,,\ X_{2A} = X_{2B} = X_{2C} \qquad (3)$$

(where $X_2$ is now a public good).

For any slope $l$ in the overlap,

$$l_A + l_B = l_C, \qquad (4)$$

where $l_C$ is the price of the public good (in terms of the private good) and $l_A$, $l_B$ the personalized prices (shares), and/or marginal rates of substitution for A and B, respectively. In the standard use of the exchange box, moving from a point on the total budget line to another point implies a new box with different dimensions and one origin shifted. A change in price will leave both the dimensions and the origins of the exchange box unaffected as long as the new budget line passes through the same point at the origin ($0_B$) (The community basket of goods). In the public good case, however, the exchange box is fixed by the slope of the budget line (price) and all $X_1$, $X_2$ combinations supported by that given budget equation.

In the private goods case, we seek to solve for $X_{1A}$, $X_{1B}$, $X_{2A}$, $X_{2B}$ under varying prices, by fixing $X_{1C}$ and $X_{2C}$. In the public good case, we solve for $X_{1A}$ and $X_{1B}$ and $X_2$ ($=X_{2A} = X_{2B}$). Since it will be premptive to fix $X_2$, $X_1$ must also vary. Thus, both goods private, the total budget equation becomes:

$$(X_{1A} + X_{1B}) + P_2/P_1\ (X_{2A} + X_{1B}) = Y/P_1, \qquad (5)$$

and we solve for $X_{1K}$ by fixing the quantities of $X_1$ and $X_2$ and letting prices and the distribution of $X_1$ and $X_2$ between A and B vary. In the public good case, the total budget becomes:

$$(X_{1A} + X_{1B}) + [(P_{2A} + P_{2B}) / P_1]\ X_2 = Y/P_1, \qquad (6)$$

and we solve for $X_{1A}$, $X_{1B}$, $X_1$, $X_2$ and $P_{2A}$, $P_{2B}$, by fixing prices ($P_2/P_1$). That is, with two private goods, for a given *fixed* quantity of $X_1$ and $X_2$, an exchange box can be defined that will accommodate an infinite number of feasible prices. In the public good case, for a given fixed price, an exchange box can be defined that will accommodate an infinite number of $X_1$, $X_2$ combinations that are implied by the total budget equation. This symmetry is of course the geometric expression of a specific aspect of the more general symmetry expressed in the exchange of roles of slope and quantity in the private-private and private-public commodity space.

In Figure 1 we describe an exchange box which complies with the equilibrium conditions when $X_1$ and $X_2$ are the private and public good, respectively. For individual A the origin is at $0_A$, with both axes and direction of preferences pointing the usual way. Initial income $0_AK$ in terms of $X_1$, and the acquisition price determined by the slope KM, describe the consumption opportunities of A in isolation. With individual B's income amounting to $0_BK$ and $X_2$ a public good, the range of possible expansion of the consumption set, at all share arrangements for A is determined by $KLN0_A$. Along KL, $X_2$ is a free good for A, so that his share $P_{2A}$ is zero. Along KM, A pays the full acquisition price for $X_2$ and his share $P_{2A}$ is equal to one.

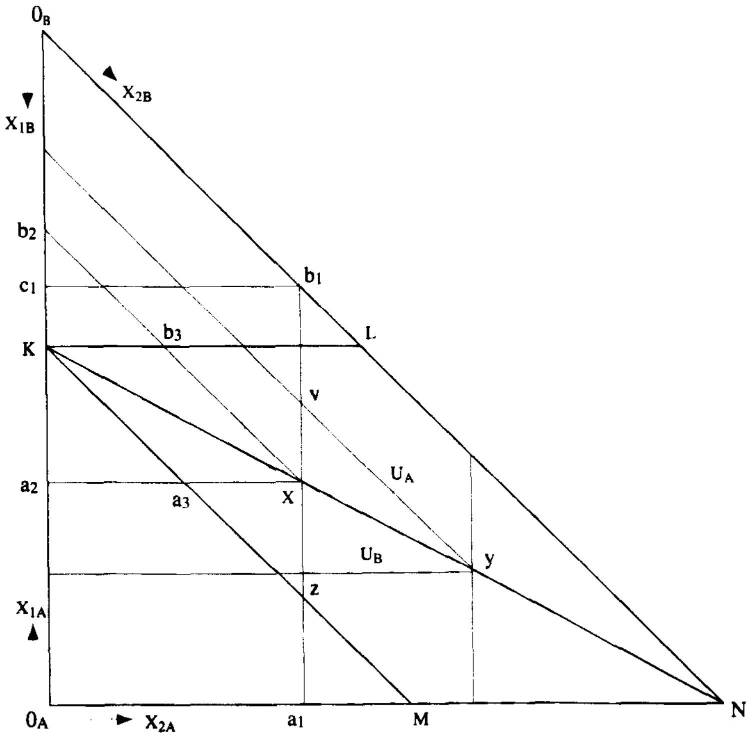


**Figure 1**

For individual B, the origin is at $0_B$ and his coordinate system is defined by the axes $0_B0_A$ and $0_BN$. (The axis $0_BN$ is always made to coincide with the total (community) budget line $0_BN$ drawn with respect to origin $0_A$). Along $0_B0_A$ we measure B's quantities of $X_1$ and along $0_BN$ B's quantities of $X_2$. The scale of $0_BN$ derives by vertical projestion from $0_AN$. Point x for example, determines $0_Bb_2$ of $X_1$ and $0_Bb_1$ of $X_2$ for B, with $0_Aa_2$ of $X_1$ and $0_Aa_1$ of $X_2$ for A. (Note that $0_Bb_1 = 0_Aa_1 = b_1C_1$ so that the quantity of the public good $X_2$ consumed by each individual and the community is the same). On the other hand, for the private good $0_Aa_2 + 0_Bb_2 = 0_AC_1$, and thus the sum of $X_1$ consumed by the two individuals is equal to the amount consumed by the community, as determined by point $b_1$.

The community consumption set and budget line be expressed in terms coordinate system. From $0_A$ the total budget line is $0_BN$ and from $0_B$ the total budget line is described by $0_AN$. For any slope of a given line such as xy we can define:

$$l_A = xz/zy,\ l_B = vx/vy \text{ and } l_C = vz/zy \qquad (7)$$

For $l_A + l_B = l_C$ to hold, $xz/zy + xv/vy = vz/zy$, and since $zy = vy$ by construction, then $zx + xv = zv$. This slope condition holds for any line inside $0_A0_BN$, and the condition holds similarly from the $0_B$ origin[2]. With the conditions holding for any point and any slope inside or on the boundary of $0_A0_BN$, the superimposition of preferences for the two individuals *drawn in respect to corresponding coordinates* will make it possible to define the Pareto efficiency locus. At any point along the efficiency locus such as X in Figure 1, the common tangency will define a pair of personalized prices which add up to the total (fixed) acquisition price, a pair of marginal rates of substitution equal to their respective personalized prices, a common level of the public good and quantities of $X_{1A}$ and $X_{1B}$ at a welfare level that will be the maximum attainable for one individual, given the welfare level for the other.

Figure 2 depicts the way by which the exchange box of Figure 1 was derived from the production set with given common technology and different initial factor endowments. The assumptions implied by the way Figure 2 is

2. It is possible to describe the individual condition in terms of tax shares and tax rates. If x is the consumption point, individual A will have a tax-share of $aa_3/a_2x$, and an income tax rate defined by the vector $0_Aa_3$ (not drawn). Individual B's tax-share will be $b_2b_3/b_2x$ and the tax rate is depicted by vector $0_Bb_3$. The two tax-shares will always add up to one while the sum of the two tax vectors seen from a common coordinate system will add up to the community ta rate $0_Ab_1$ $(0_Ba_1)$.

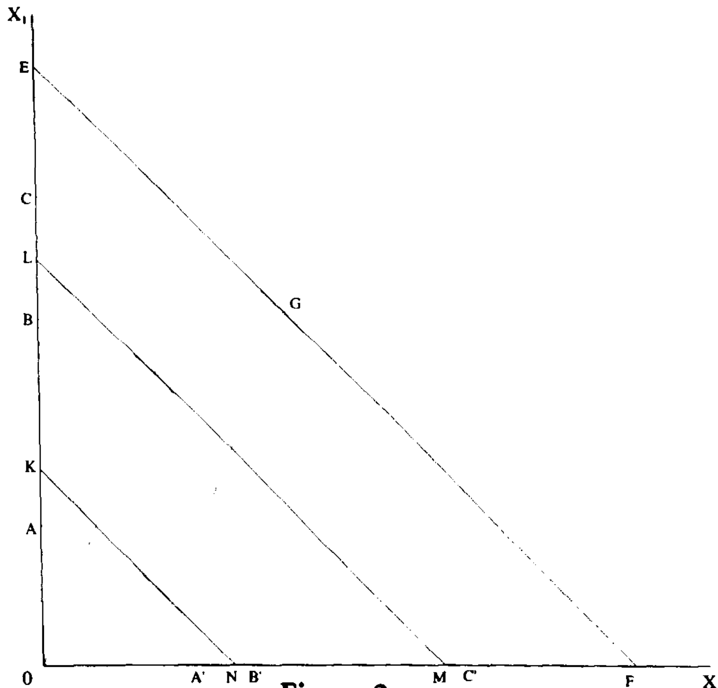

**Figure 2**

drawn are (a) that each individual is identified with production activity rather than with a single factor of production, and (b) that the existence of a large outside market makes it possible to separate the production from the consumption points for the community and each individual. So with the outside price 0E/0F fixed for the community and for each individual, A will produce on $g_A$ and have an opportunity set 0KN; B on $g_B$ with consumption opportunity 0LM and the community on G with 0EF. The community and the individual amounts of $X_{1A}$, $X_{1B}$, $X_1$ and $X_2$ will be determined, along with the amount and the composition of trade with the outside [3]. The production blocks in Figure 2 will generate an infinite number of relative incomes and exchange configurations (Edgeworth boxes), each defined by a different point on the production block CC' and the corresponding points on AA' and BB'.

3. Note that in case of a "closed" community, the meaning of intra-community trade becomes problematic, with only one private and one public good and the "peculiarities" in the consumption of the public good might penetrate the production sphere in the case of a "closed" community. We will not, however, consider this problem any further here.

### III. Certain Problems with Shibata's Analysis

The geometry developed in II, along with the addition of preferences for A and B shed some light on the discussion about the existence and the nature of mechanisms that might ensure Pareto optimality. In this respect, Shibata's attempt to draw analogies between bilateral monopoly and equilibrium processes in an economic environment with public goods represents an important contribution. The possibility, however, that this contribution will be usefully incorporated into the analysis of public goods is seriously undermined by a succession of errors that are particularly damaging when the case of increasing marginal costs is considered. The sources of errors in Shibata's anlysis are: (1) the failure to understand the nature of the symmetry between private-private and private-public commodity spaces as it manifests itself in the Edgeworth geometry, and (2) a procrustian zeal to fit together sets that, by definition, can not fit together.

In the first case, the transformation rule Shibata uses to transform B's budg-

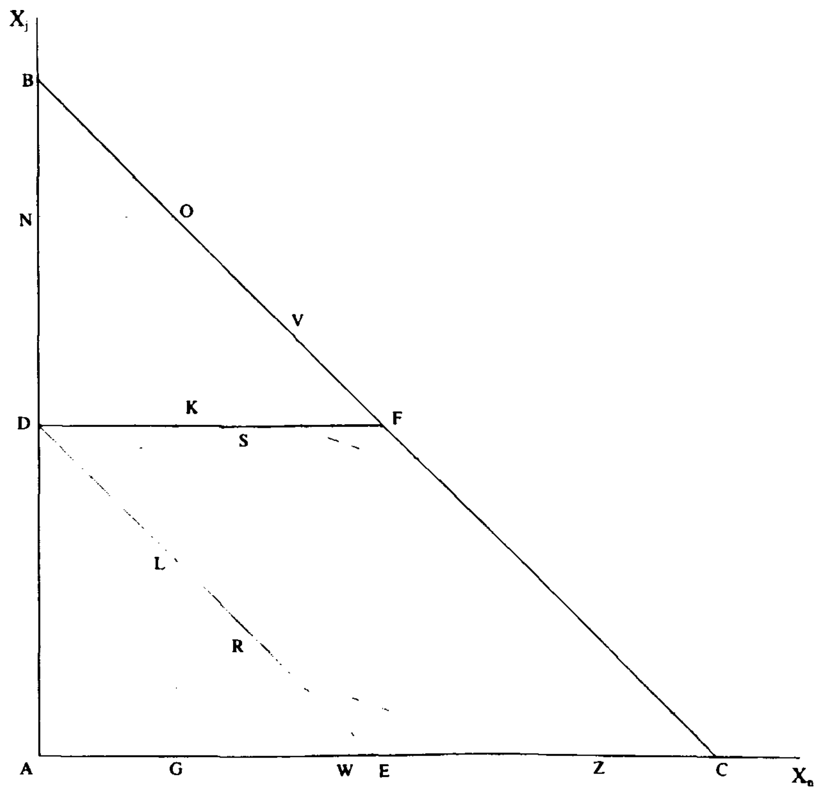


**Figure 3**

et and preferences is, in the light of our description, unnecessary but nevertheless, its application has no adverse effect on the analysis as long as it is used with a linear transformation curve (fixed price). In Figure 3 (which incorporates Shibata's figure 4) it is not clear which are the proper axes for B. Quantities of $X_n$ (the public good) for B are indicated along the AC axis. For the private good $X_j$ that goes to B, however, there is no clear instruction as to how it should be interpreted. From later comments by Shibata (p.24) any of DN or K0 or J0 is implied. DN and K0 will be, in general, incorrect (only producing in isolation will make them correct); J0 on the other hand will always give the right quantity for a given price by construction, since it is right side of the parallelogram formed by B's coordinates and proper axes. So despite inadequate reasoning and a transformation procedure the purpose of which is not well understood, as well as a vaguely defined coordinate system, the geometry coincidentally produces the correct results. Many problems ensue, however, as soon as this reasoning is extended to accommodate nonlinear production possibility curves.

Figure 4 reproduces Shibata's figure 8. Set ABC in the figure is, according to Shibata, an Edgeworth box with one private and one public good in the presence of increasing marginal costs. AD and DB describe the initial income

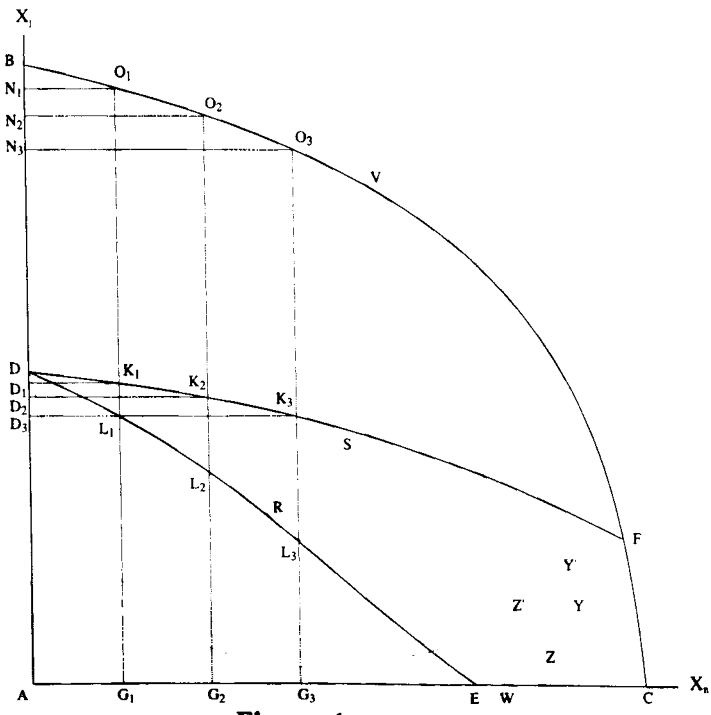


**Figure 4**

distribution in terms of $X_j$; DE and DF are the production-consumption possibility curves in isolation and BD the community transformation curve. Curves such as SZ' are B's indifference curves, and VW, the tangency locus between A's and B's indifference curves, defines the contract curve. BF maintains the convexity of B's production block *and* exactly coincides with part of the community production block. DE is A's production block *and* precisely coincides with the upper part of the community production block, when D is placed on B.

Contrary to Shibata, ABC is not an Edgeworth box. Rather, it is a summary of production and consumption information sufficient to construct an infinite number of Edgeworth boxes, each corresponding to a different rate of transformation (given that the community described can trade with the outside without affecting prices). AD and DB do not represent the initial income distribution (in both relative and absolute terms) but only one possible distribution out of an infinite number of such distributions described by the production frontiers BC, DE, and DF. Also DE and DF cannot be the production-consumption blocks of A and B, respectively, for if they are, then BC cannot be the resulting community production block. Were this true, then any of the individual production blocks placed on the other with axes properly aligned would be tangent to BC only at one point. The construction of ABC as described is therefore logically impossible.

Preferences for B described by SZ' cannot be defined until a coordinate system has been defined, and such a system, in turn, can be defined only with respect to a given point on BC, and a price line tangent to that point. The contract curve, VW, cannot be defined except in the case of a given price, just as a contract in the private-private commodity space can be defined only for a given quantity of the two commodities. DF cannot be B's production-consumption possibility curve even for a given point on BC, unless it is made concave to the origin B in order to reflect increasing marginal costs.

Points $O_1$, $O_2$, $O_3$ in our Figure 4 (Shibata's figure 8) cannot be community production-consumption points unless efficiency in production and all budget constraints are ignored. It should be noted here that conditions for efficiency in production need not be altered because of publicness in consumption.

All of the errors in section III of Shibata's article do not derive from the faulty construction of the Edgeworth box, however. Consider, for example, his description in reference to his figure 9 (Figure 4 here). The independently attained production-consumption box for B is DBF, for A is DAE. The sum of these two must equal ABC in order for the box-cum-transformation curve to work. But ABD obviously cannot be the vector sum of ADE and BFD. The ratios $AD_1/AD$, etc., are not income distribution ratios. These could, however, be $X_{jA}/X_{jB}$ ratios, given that A produces on his production frontier at a point implied by $AG_1$, and B specializes in $X_j$, with different tranformation rates and factor prices.

To recapitulate:

1. There is no way to describe one Edgeworth box from the total information of production possibility frontiers. Insistence in such derivations cause a series of other errors to be committed. Viz:

a) All relevant information from either the individual tranformation curves or the community's tranformation curve or both, is distorted beyond any usefulness.

b) The preference map for B cannot be defined under the circumstances, and similarly,

c) The production set for B cannot defined.

2. Ignoring error 1) for the present, the definition of absolute and relative incomes is wrong. For example, $AD_3/AB$ is not A's relative income. His income at that level of $X_j$ will be determined by the $X_j$ intercept of the tangent on A's tranformation curve at the point determined by $D_3$ of $X_j$, and the corresponding amount of $X_n$ (not shown in the diagram).

Further, consider Shibata's statement that "... depending upon the given state of distribution of ownership of factors of production and the given technological condition, the income-redistribution effect may become so unfavorable to one party that, even if the party bears the entire tax bill, production of a public good will place the unfavorably affected party on an indifference curve lower than its original one" (1971, p. 26). Even allowing for the errors committed above, there is still insufficient analytical support for Shibata's "proof" of this proposition. Let us examine the implications of Shibata's figure 9 (our Figure 5) in this context. Producing and consuming indepedently, A will specialize in the production and consumption of the private good $X_j$ at D. Upon community formation, and before the publicness of $X_n$ is considered, B produces and consumes at point S. That will change factor income by changing output prices, because the community is a closed economy.

In order, however, for any interaction to take place between A and B, output price as implied by the slope at D must be different from output price implied at S. Also the prices of the public good ($X_n$) must be higher at S that it is at D, if any income redistribution effects are to occur in the direction stated in Shibata's proposition upon community formation. B's demand for the public good will increase the price of $X_n$ facing A, inducing A to shift his production point from D to a point closer to E along his production possibility frontier. At the same time, the possibility of trade with B will enhance A's consumption opportunity set. In isolation, both individual sets were defined by their respective production possibility sets, so that production for each individual was identical with that individual's consumption. With the opening of trade between A and B - a necessity in Shibata's example if redistribution is to take place - the new consumption opportunity set for A will be defined by the targent at his new production point. A's new consumption point, given his preferences as drawn in Shibata's figure 9, will be somewhere higher than D along the $X_j$ axis. A, confronted with a price higher than his subjective evaluation for $X_n$, will find it to his benefit to shift some of his resources to produce some Xn in order to exchange it for $X_j$, and thus reach an indifference curve

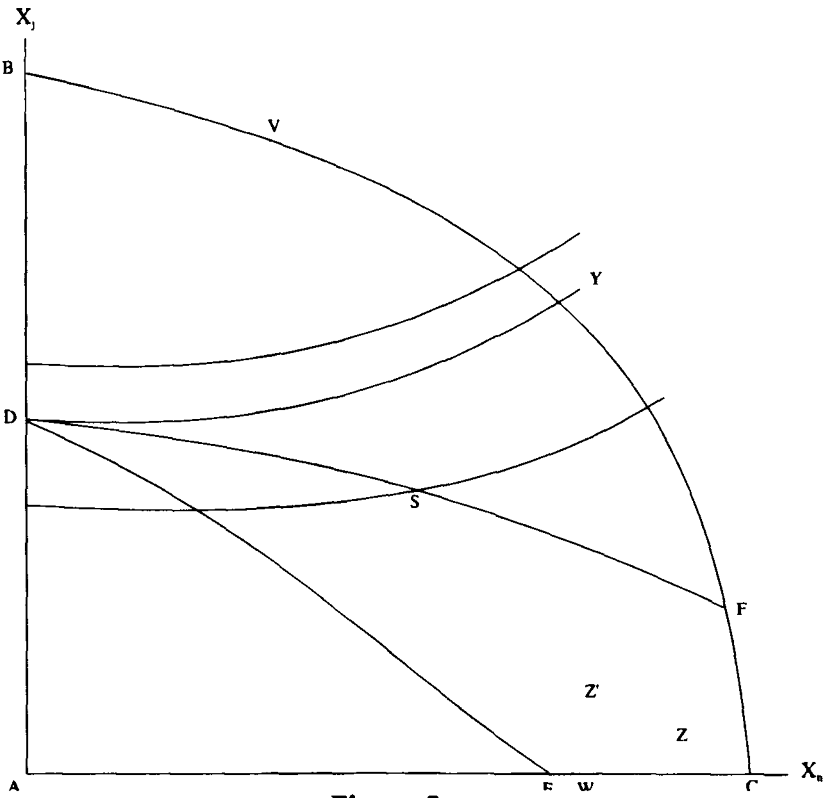


**Figure 5**

higher than DY. If, in a two-commodity market, the publicness of $X_n$ makes exchange of final goods problematic, factor mobility - through lending or borrowing of a factor at the going reward rate - will achieve the same result (Krauss and Johnson, 1975). With publicness in comsumption of $X_n$ and with A's share equal to zero, his new consumption point (S) will be at the intersection of the horizontal line, starting somewhere above D, and the vertical line passing from S. At such point, individual A will be strictly better off compared to his position at the independently attainable production-consumption opportunity set. His total gains can be decomposed into gains from trade and gains from publicness in the consumption of $X_n$. Had this line of analysis been followed correctly by Shibata, it would have rendered the faulty construction of the Edgeworth box plainly obvious. Inasmuch as the point of consumption (S) for A and (S) for B would not have coincided, as a result, the community's consumption point would appear to be in two places at the same time.

## IV. The impossibility of an Immiserizing Effect of a Freely Provided Public Good

As shown above, Shibata's propositions and conclusions in his Section III are without any analytical support. The lack of such analytical support, however, is not by itself positive proof that these propositions are wrong. There might be, for example, conditions under which a freely provided public good could place the recipient on a lower indifference curve due to offsetting adverse income or redistribution effects. In order to investigate such a possibility, we shall use the geometry developed in section II of this paper.

Assume two individual with identical production functions and different ratios of factor endowments, producing and consuming two goods $X_1$, a private good and $X_2$, a public good. There are two states to be compared here, one in which both individuals produce and consume independently of each other, so that $X_2$ is viewed by each as a private good. In a second state the two individuals, realizing the public nature of $X_2$, from a community[4]. Prices of

4. "Community formation" is used here to mean agreement on a set of rules and institutions in order to capture and divide the gains from the publicness in consumption of $X_2$. Because the merging is voluntary, the collectively decided amount of X will never be lower than any of the two individuals' indepentently attainable optima at the prevailing production price. If that were the case, the individual who demands more than the collectively provided amount of X would find it possible and desirable to increase the quantity of the public good unilaterally, by assuming the total cost of the incremental change until the quantity of the public good is equal to the quantity he demands along his independently attainable budget line.

final goods and factor prices are equalized between A and B by trade with an outside market large enough to hold prices facing A and B constant. If alternatively, A and B form a closed economy, factor prices are equalized either by exchange of final goods, before community formation, or by free mobility of factors between production plants A and B, after community formation.

Figure 6 describes *the community* as *an open economy*, with $0_A$ and $0_B$ the origins for A and B, respectively. The community's origin coincides with $0_A$ and the production transformation curves are drawn as $T_AT_A$, $T_BT_B$, and $T_CT_C$. The prevailing price determined outside is reflected in the slopes ($0_BG$, FE and FK) of the budget lines for the community and the two individuals, measured from their respective origins. The community production point, $C_p$, is the vector sum of A's production point $A_p$, and B's production point, $B_p$, taken from origin $0_A$. Similarly, the community consumption point $C_c$ is the vector sum of the individual consumptions points $A_C$ and $B_C$. The way in which the tranformation curves are drawn implies that A possesses more, relative to B, of the factor used intensively in the production of the private good $X_1$.

The community as a whole is a net importer of $X_2$ as shown by the relative positions of $C_p$ and $C_c$. A is an importer of $X_1$ and B an importer of $X_2$ and, as shown, the net sum of individual amounts of trade equals the community amount of trade with the outside. The amount of the public good available for collective consumption is not necessarily the sum of the amounts the individuals decide to produce, but the amount that the community, through an unspecified mechanism, decides to consume. The independence of consumption and production can be assured only by assuming that each individual can always produce and sell to the outside any amount of the public good at the prevailing price.

In Figure 6, H is the collective production point as seen from origins $0_A$ and $0_B$; similarly, J is the collective consumption point at the moment of community formation and before any adjustment in the consumption points of A and B has been made. Consumption points $A_C$ and $B_C$, therefore, were determined when $X_2$ was consumed independently. With the price fixed from the outside, production points $A_p$, $B_p$, and $C_p$ will not change, and any changes in the consumption points prompted by the collective consumption of $X_2$ will be facilitated by changes in the trading positions of the individuals and the community. Thus, the assumption of an outside trading partner enables us to fix the community's Edgeworth box and, effectively to isolate questions of production efficiency from those of public consumption.

By construction, as described in Section II, the slope of any line *l* inside

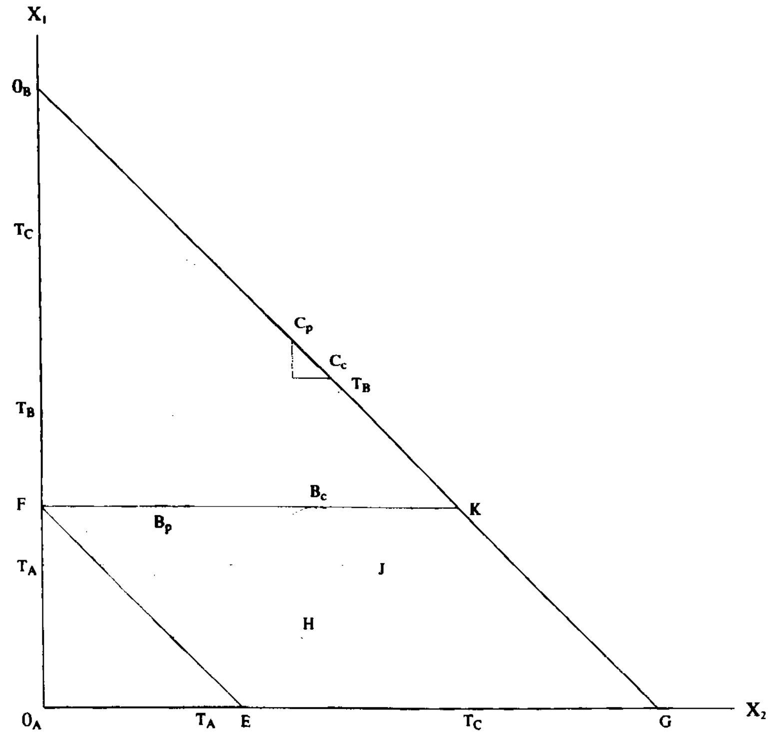


**Figure 6**

the box must obey the relationship $l_A + l_B = l_C$. Hence, when a line has a slope, say $l_B = l_C$, in order for the condition to hold then $l_A = 0$. That is to say, when a line such as FK is drawn in respect to origin $0_B$, equal in slope to the prevailing price, it will always be read by $0_A$ as having a slope equal to zero. Correspondingly, a line such as FE drawn to reflect the common price will be read by $0_B$ as having slope zero. The implications of this relationship (which will always hold by virtue of the construction rules used) is that moving along FK from F to K will increase consumption of $X_{2B}$ at the expense of consumption of $X_{1B}$ at the rate equal to the acquisition (production) price, while at the same time, it will increase consumption of $X_{2A}$ (where $X_{2A} = X_{2B} = X_2$) at not cost in terms of $X_{1A}$ for A. The analogous situation, with A's and B's positions reversed, is described by movements along FE.

The important point for the proposition investigated is that, by definition, the independently attainable consumption frontier for B will always be a horizontal line (slope zero from origin $0_A$), starting from F where A's budget line cuts the $X_1$ axis.

Suppose now that B for some reason undertakes to pay the full price of the public good. This means that the common consumption point will be somewhere along FK, for instance $B_C$. At this point, B will attain the same welfare level he had before community formation, while A will consume $0_AF$ of $X_1$, and $FB_C$ of $X_2$ which is provided freely to him. There is no combination of preferences that will make A worse off in comparison with his pre-community consumption point, given that his new comsumption point will be at the zero price (share) line FK, that some $X_2$ is paid by B, and $X_2$ is a positive good for both A and B. The worst possible position for A, at the limit, will be when B decides to consume at F with zero $X_{2A}$, in which case A will consume at $A_C$, his independently attainable optimum. Nothing that B can do by means if changing his position along his independently attainable budget line, will cause A to consume at a position lower than $A_C$.

This open economy example is not the situation implied by Shibata's analysis, however. In his example, prices and production points, factor rewards and relative incomes, change as a result of changes in the demand for $X_1$ and $X_2$. This, of course, describes the community as a closed economy, and in this case whatever amount of $X_2$ is produced in the community by both producers, is automatically available for collective consumption. In Figure 7, we describe A's transformation curve, $T_AT_A$, and his preferences. Because a community production point and, therfore, a price cannot be fixed, we are unable to describe an Edgeworth box in this situation. It is possible to carry out the analysis, however, considering only A's origin, for it is the fixed part of any constructible box, and since B's consumption has been assumed limited along his independently attainable budget line, which will always have, for A, slope zero.

There are two ways to define the independently attainable consumption point that will provide the reference base. Point $A_1$ where A produces and consumes in isolation with different implied commodity and factor prices, or point $A_3$ where A trades with B at equalized prices but does not consume collectively with B as yet.

We must keep carefully separate the parallel developments in production and consumption spheres. Shibata's proposition is phrased to imply changes in A's production as well as consumption points which will result in charges in factor prices. This, in turn, implies that both A and B are isolated from any outside markets. Such price changes will occur because A and B started trading and formed community for collective consumption, or alternatively because they formed a community for collective consumption without any trade between them. In the former case, with trade possible and with prevailing prices, after trade opening, denoted by the slope FH, A will go from pro-

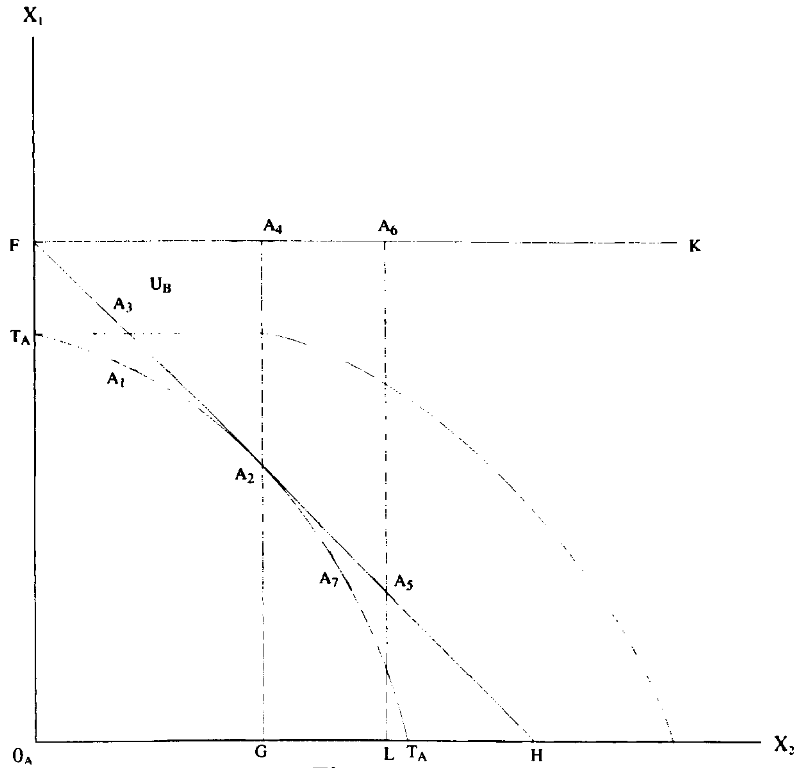


**Figure 7**

ducing and consuming at $A_1$ to producing at $A_2$ and consuming at $A_3$. The opening of trade will imply gains for A, or at the limit no losses, if prices remain the same before and after trade opening (Samuelson, 1962). If the meaning of exchange in final goods becomes blurred by the presence of the public good, mobility of the primary goods between the two production plants at the prevailing factor prices will equalize commodity prices and ensure efficiency in production as shown by Mundell (1957).

In order to facilitate the derscription and keep production and consumption separate, we can assign a community clearing house as part of the community formation. A and B produce at a point determined by the price implicit in the community combination ($X_1$, $X_2$) that will be decided by the commonly accepted collective decision mechanism[5].

5. We are not implying that the finding of a collective decision mechanism is an easy feat to perform. We have simplified the problem in the present instance by assuming away half of the

If FH describes the price corresponding to the final community production point, A and B produce according to this price and turn the total quantity of $X_2$ into the clearing house, which in turn credits each producer with their corresponding amount of $X_2$ in terms of $X_1$. The total amount of $X_2$ is then made available for collective consumption; the clearing house collects taxes according to assigned shares and pays back according to amounts produced in terms of $X_1$, the private good.

If A's preferences are decribed by the indifference curves tangent to $A_1$ and $A_3$, and his production point by $A_2$, then community public good will be *at least* $0_AG$, and A will be able to consume at $A_4$ (at least) with $0_AF$ of $X_1$ and $0_AG$ of $X_2$. It is clear that $A_4$ is strictly superior to both $A_1$ and $A_3$.

If A's preferences are described by tangencies at $A_7$ and $A_5$ with production as before at $A_2$, then the community will provide at least $0_AL$ of $X_2$ and A will be able to consume at $A_6$ with $0_AF$ of $X_1$ and $0_AL$ of $X_2$. This position is strictly better than either $A_7$ or $A_5$. Any adjustment at $A_4$ or $A_6$, if possible, might improve the position of both A and B as shown by the intersecting indifference curves at $A_4$; such adjustments, however, will strengthen the conclusions with respect to A's relative improvements.

The last example concerns a less likely, but nevertheless possible situation, where the two individuals interact only by collective consumption of $X_2$, with no possibility for exchange of either final or primary goods. In this case, each individual's consumption set is indentical to his respective production set. Prices and factor rewards remain unaffected and presumably unequal.

Before community formation, A will produce and consume at a point such as $A_1$ or $A_7$, depending on his preferences. If B pays the total cost of an amount of $X_2$, say $0_AG$, then A's consumption set will change to $T_AT'_A$, and A will consume $0_AT_A$ of $X_1$ and at least $0_AG$ of $X_2$. In any case, whatever the freely provided amount of $X_2$, and whatever A's consumption intensity, his position will be improved with respect to his pre-community consumption point, for his enhanced consumption-production set $T_AT'_A$ contains his original set $T_AT_A$, except at point $T_A$ on $X_1$, where no $X_2$ is produced or freely provided.

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problem by fixing individual tax shares ($S_A = 0$, $S_B = 1$). The other half is circumvented by considering any arbitrary quantity of the public good that can result from any unspecified collective decision process which is positively responsive to individual preferences.

## V. Concluding Comments

The main theme in this paper concerns the symmetry between private-private and public-private commodity spaces. Particular attention is paid to the geometric aspects of this symmetry in simple general equilibrium models, and a precise analogue of the standard Edgeworth box is developed for a two-individual community with one public and one private good. The minimal definition of the exchange box as an overlap of three interdependent coordinate systems, constructed to satisfy certain conditions about coordinates (quantities) and slopes (prices, marginal rates of substitution), provides the basis for the generalization of the standard Edgeworth box.

The understanding gained from this generalization is then used to critically examine Shibata's earlier attempt to adopt the Edgeworth box for public goods and to point out certain defects in his analysis. Additionally, the same geometry is appilied to demonstrate the impossibility of immiserization through free provision of a public good, thus negating Shibata's assertion to the contrary.

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