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A set of Economic Equations and a Generalisation of Brouwer's Fixed Point Theorem*

by

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(English translation by Alekos Tsitsovits)

The object of this notice is the solution of a typical economic equations set. This set has the following properties:

(1) The goods are produced not only from the so-called "natural production factors" but mainly from *each other*. Specifically the production processes may by *cyclic*, i.e. good G_1 is produced by means of good G_2 and G_2 by means of G_1.

(2) Under certain circumstances there may exist more technically possible production processes than goods. The usual method of "equation counting" is therewith inefficient. Decisive is indeed to find out which processes are really used and which (being "non profitable") are not.

In order to discuss (1) and (2) in pure form we shall generally idealise some other elements of the situation (cc. §§1 to 2). Most of these idealisations are not substantial, but we are not going into further details here. Our problem setting leads convincingly to a set of inequalities (3)-(8′) in §3, of which the feasibility to solve is not at all evident, i.e. *it can not be proven by any qualitative argumentation*. On the contrary the mathematical proof is successful only by means of a generalisation of Brouwer's fixed point theorem, i.e. by using really deep laying *topologic* facts. This generalized fixed point theorem (the "theorem" of §7) is also interesting on its own.

The connection to topology may on first sight be surprising indeed, but the author thinks this is natural by this kind of problems. It is directly caused

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* Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, in: *Ergebnisse Eines Mathematischen Kolloquiums*, unter mitwirkung von F. Alt, K. Gödel, A. Wald, herausgegeben von Karl Menger, Wien, Helt 8, pp. 73-83, 1935-1936, Leipzig und Wien Franz Deuticke, 1937.

by the appearance of a certain "minimax" problem, well known from variation calculus. In our problem this "minimax" problem is formulated in §5. It is closely keen to another which appears in the theory of social games [cc. 2) in §6].

A direct interpretation of the here resulting function $\Phi(X,Y)$ would be very convenient. Its role appears to be similar to the role of thermodynamic potentials in phenomenological thermodynamics and it will have presumably a similar role even in the case of phenomenological generality (independently of our unnaturally constraining idealisations).

Another feature temporarily not integrated in our theory is the remarkable duality (symmetry) of the monetary variables (prices y_j, interest factor β) and the technical ones (production intensities x_i, economy expansion coefficient α) This duality is extremely noticeable in §3 (3)-(8′) as well as in §4 (7*)-(8*) and also in the "Minimax" – formulation of §5 (7**)-(8**).

Finally, attention is due to the results of §11, from where it can be concluded amongst other that (if our assumptions are valid) the normal price mechanism results to the purely technically most expedient distribution of production intensities. Since we have excluded all monetary complications, this is not unreasonable.

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The following considerations have been presented for the first time in winter 1932 at the mathematical colloquium at Princeton University. The reason for their present publication refers to an invitation by Mr. K. Menger, to whom the author also here expresses his thanks.

1. Consider the following problem: There exist n goods $G_1, \ldots, G_n$, which can be produced by m processes $P_1, \ldots, P_m$. Which processes will be used (as "profitable") and which prices for the goods will be valid? The problem is obviously non-trivial, because each of its halfs can only be answered if the other already is – i.e. it is implicit. We remark in particular:
   (a) Since it can be $m>n$ it can certainly not be solved by the otherwise usual method of "equation counting".
   In order to exclude another kind of complications we assume that:
   (b) The amount of production is constant – And:

(c) The natural production factors, inclusive labour, are at unlimited disposal.

The important phenomenon which we wish to grasp is this: The goods are produced from each other through the production processes (s. equation (7)) and we want to find out, (i) which processes will be used and with which intensities, (ii) the relative speed of growth for the total goods quantity, (iii) which prices will be established (iv) which rate of interest is valid. In order to isolate completely this phenomenon, we assume further:

(d) The only existing consumption is the consumption of goods in the production processes, including necessarily the life sustaining goods consumption of labourers and employees, i.e. we assume that every income over the life sustaining minimum is completely reinvested.

2. Each process P_i, $i=1, ..., m$, is of following nature: It uses the quantities a_{ij} (measured in arbitrary units) of corresponding goods G_j $(j=1, ..., n)$ and produces the quantities b_{ij} of the same. It can so be formulated symbolically:

$$(1) \qquad P_I : \sum_{j=1}^{n} a_{ij} G_j \rightarrow \sum_{i=1}^{n} b_{ij} G_j$$

Where it must be noticed that:

(e) Capital goods have to be considered simply on both sides of (1). The wear of a capital good has to be described by introducing its various wear phases as separate goods and considering these separately for each P_i:

(f) Each process P_i has as time term the time unit. Longer processes have to be divided into partial processes of this length, introducing if necessary, the intermediate products as special products.

(g) (1) can in particular describe the case where a good G_j can only be produced together with certain other goods, its permanent by-products.

In the real process of the whole economy these processes P_i, $i=1, ..., m$ are used with certain intensities x_i, $i=1, ..., m$. I.e. for the total process the quantitative data in equation (1) have to be multiplied by x_i. We write in symbols:

(2) $$W=\sum_{i=1}^{m} x_i P_i$$

$x_i=0$ means that process P_i remains unused.

We are interested in the situations of the whole economy where it expands without changing its structure, i.e. where the relations between the intensities $x_1: \dots :x_m$ remain unchanged but $x_1, \dots, x_m$ themselves are allowed to change. Then $x_1, \dots, x_m$ are multiplied by a common factor α per time unit. This factor α is the *expansion coefficient of the whole economy*.

3. The numerical unknowns in our problem are:
   (i) The intensities $x_1, \dots, x_m$ of processes $P_1, \dots, P_m$,
   (ii) The expansion coefficient of the whole economy α,
   (iii) The prices $y_1, \dots, y_n$ of the goods $G_1, \dots, G_n$,
   (iv) The interest factor β ($\beta=1+\frac{z}{100}$, z is the interest rate per time unit).

   Obviously it is always

(3) $$x_i \geqq 0$$ (4) $$y_i \geqq 0$$

and, because a solution with $x_1= \dots =x_m=0$ or $y_1= \dots =y_n=0$ were meaningless

(5) $$\sum_{i=1}^{m} x_i > 0$$ (6) $$\sum_{j=1}^{n} y_j > 0$$

The economy equations are herewith:

(7) $$\alpha \sum_{i=1}^{m} a_{ij} x_i \leqq \sum_{j=1}^{m} b_{ij} x_j$$

(7′) and in the case of strict inequality in (7) it is $y_j=0$.

(8) $$\beta \sum_{j=1}^{n} a_{ij} y_j \geqq \sum_{i=1}^{n} b_{ij} y_j$$

(8′) and in the case of strict inequality in (8) it is $x_i=0$.

(7), (7′) mean: The quantity of a good G_j can consumed in the total process (2) cannot be greater than the quantity produced. If though the consumption is less, i.e. there is a G_j overproduction, then G_j becomes a free good and its price y_j will become 0. And (8), (8′) mean: In the equilibrium situation a profit cannot be extracted in any process P_i (because then the prices or the interest rate will rise – it is clear how to understand these idealizations). But, if there is a loss, i.e. P_i is unprofitable, then P_i will stay unused, its intensity x_i becoming 0.

Coefficients a_{ij}, b_{ij} have to be considered as fixed quantities whereas x_i, α, y_i, β are the unknowns. There are m+n+2 unknowns, because though only the relations in x_i, y_j, $x_1 : \dots : x_m$, $y_1 : \dots : y_n$ are of importance there are really only m+n.

Corresponding there are m+n constraints (7)+(7′) and (8)+(8′). Because though these are not equalities but rather complicated inequalities, the equality of these numbers does by no means at all ensure the solution of the equations set.

The dual symmetry of equations (3), (5), (7), (7′), in variables x_i, α and of the term "unused process" on the one hand and of equations (4), (6), (8), (8′), in variables y_j, β and of the term "free good" on the other, seams to be remarkable.

4. Our aim is to solve (3)-(8′). We stall prove that: *There exist always a solution for (3)-(8′).* There can indeed exist several solutions with different $x_1 : \dots : x_m$ or $y_1 : \dots y_n$. In the first case it is possible because we have not excluded the case where several P_i describe the same process or that a certain P_i results as a combination of others. In the second case it is possible because some goods G_j appear, possibly in every process P_i in a fixed relation to some others. But even if these trivial cases are excluded, there exist several solutions $x_1 : \dots : x_m$, $y_1 : \dots : y_n$ because of less direct reasons. In contrary it is of importance that α, β have in all solutions the same value. I.e.

α, β are uniquely determined

We shall see indeed that α and β can be directly characterised in a simple way (s. §§10-11). In order to simplify our considerations we assume that always

(9) $a_{ij} + b_{ij} > 0$

(of course it is always a_{ij}, $b_{ij} \geqq 0$). Because a_{ij}, b_{ij} can be arbitrarily small this constraint is not very severe. It is though necessary in order to ensure the uniqueness of α, β, because otherwise W could dissolve into unconnected parts. These questions will be nevertheless examined on another occasion.

Let us now consider an (hypothetical) solution x_i, α, y_j, β of (3)-(8′).

If there were in (7) always <, then because of (7′) there would be always $y_j=0$, contradicting (6). If there were in (8) always >0, then because of (8′) there would be always $x_i=0$, contradicting (5). So: in (7) is always $\leqq$, but at least one =, in (8) is always $\geqq$, but at least once =. Therefore:

(10) $$\alpha = \operatorname*{Min}_{j=1,\dots,n}\left[\sum_{i=1}^{m} b_{ij}x_i \Big/ \sum_{i=1}^{m} a_{ij}x_i\right]$$

(11) $$\beta = \operatorname*{Max}_{j=1,\dots,m}\left[\sum_{j=1}^{n} b_{ij}y_j \Big/ \sum_{j=1}^{n} a_{ij}y_j\right]$$

In this way x_i, y_j determine uniquely α, β. (The right sides of (10), (11) can never assume the meaningless form $\frac{0}{0}$ because of (3)-(6) and (9)).

We can therefore formulate (7)+(7′) and (8)+(8′) as constraints for x_i, y_j only:

(7*) For every j=1, ..., n, where

$$\sum_{i=1}^{m} b_{ij}x_i \Big/ \sum_{i=1}^{m} a_{ij}x_i$$

does not assume its minimal value (for all j=1, ..., n), it is $y_j=0$.

(8*) For every i=1,...,m, where

$$\sum_{j=1}^{n} b_{ij}y_j \Big/ \sum_{j=1}^{n} a_{ij}y_j$$

does not assume its maximal value (for all i=1,...,m), it is $x_i=0$.

(In (7*) $x_1, \dots, x_m$ have to be considered as fixed, in (8*) $y_1, \dots, y_n$). We have to solve therefore (3)-(6), (7*), (8*) referring to x_i, y_j.

5. We name X′ a series of variables $(x'_1, \ldots, x'_m)$ which fulfils the analoga of (3), (5)

(3′) $x'_i \geqq 0$ **(5′)** $\sum_{i=1}^{m} x'_i > 0$

and Y′ a series of variables $(y'_1, \ldots, y'_n)$ which fulfils the analoga of (4), (6)

(4′) $y'_j \geqq 0$ **(6′)** $\sum_{j=1}^{n} y'_j > 0$

We set also

(12) $$\Phi(X',Y') = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x'_i y'_j \Big/ \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x'_i y'_j$$

Be $X=(x_1, \ldots, x_m)$, $Y=(y_1, \ldots, y_n)$ the (hypothetical) solution, $X'=(x'_1, \ldots, x'_m)$, $Y'=(y'_1, \ldots, y'_n)$ freely variable, but in a way that (3)-(6) and (3′)-(6′) are valid, then (7*), (8*) can be formulated as follows, as easily verifiable:

(7)** $\Phi(X, Y')$ assumes by $Y'=Y$ its minimal Y′-value.

(8)** $\Phi(X', Y)$ assumes by $X'=X$ its maximal X′-value.

The question of solvability of (3)-(8′) transfers to the question of solvability of (7**), (8**) and latter can be formulated thus:
(*) *Consider $\Phi(X',Y')$ in the spaces limited by (3)-(6′). We search for a saddle point $X'=X$, $Y'=Y$ i.e. a point where $\Phi(X,Y')$ has a Y′-minimum and simultaneously $\Phi(X',Y)$ a X′-maximum.*

(7), (7*), (10), and (8), (8*), (11) result to:

$$\alpha = \sum_{j=1}^{n}\left[\sum_{i=1}^{m} b_{ij}x_i\right] y_j \Big/ \sum_{j=1}^{n}\left[\sum_{i=1}^{m} a_{ij}x_i\right] y_j = \Phi(X,Y) \quad \text{and}$$

$$\beta = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} b_{ij}y_j\right] x_i \Big/ \sum_{i=1}^{m}\left[\sum_{j=1}^{n} a_{ij}y_j\right] x_i = \Phi(X,Y)$$

I.e.:
(**) *If our problem is solvable, that is if $\Phi(X',Y')$ has a saddle point $X'=X$, $Y'=Y$ (see above), then it is*

(13) *$\alpha=\beta=\Phi(X,Y)$ = the value at the saddle point.*

6. Because $\Phi(X',Y')$ is homogenous (in X', Y', i.e. in $x'_1, \ldots, x'_m$ and $y'_1, \ldots, y'_n$) the problem is not influenced if (5′), (6′) (and correspondingly (5), (6)) are replaced through the normalisations

$$(5^*) \quad \sum_{i=1}^{m} x'_i = 1 \qquad\qquad (6^*) \quad \sum_{j=1}^{n} y'_j = 1$$

Doing this we name S the set of X' described by

$$(3') \quad x'_i \geqq 0 \qquad\qquad (5^*) \quad \sum_{i=1}^{m} x'_i = 1 \quad \text{and}$$

T the set of Y' described by

$$(4') \quad y'_j \geqq 0 \qquad\qquad (6') \quad \sum_{j=1}^{n} y'_j = 1$$

(S, T are m−1, corr. n−1 dimensional simplices).
In order to solve (*)[1] we return to the more direct formulation (7*), (8*), combined with

$$(3) \quad x_i \geqq 0 \qquad\qquad (5^*) \quad \sum_{i=1}^{m} x_i = 1$$

$$(4) \quad y_j \geqq 0 \qquad\qquad (6^*) \quad \sum_{j=1}^{n} y_j = 1$$

i.e. by this, that $X=(x_1, \ldots, x_m)$ lies in S and $Y=(y_1, \ldots, y_n)$ lies in T.

1. The solvability of our problem is curiously connected with the solvability of a problem appearing in the social games theory, with which the author has dealt elsewhere (Math. Annalen, 100, 1928, pp. 295-320, in particular pp. 305 and 307-311). That problem is a special case of (*) and is dealt with in a new way through our solution of (*) (s. further) it is indeed: For $a_{ij}=1$ it holds because of (5*), (6*) $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x'_iy'_j = 1$, and therefore $\Phi(X',Y') = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x'_iy'_j$ and therefore our (*) coincides with (op. cit. p. 307). (Our $\Phi(X',Y')$, b_{ij}, x'_i, y'_j, m, n correspond to the $h(\xi,\eta)$, α_{pq}, ξ_p, η_q, M+1, N+1 there).

It is also remarkable that (*) has not led, as usual, to a simple maximum or minimum problem, which were obviously solvable but to a saddle point or minimax problem where the question of solvability lies much deeper.

7. We shall prove a more general theorem:
Be R_m the m-dimensional space of all points $X=(x_1,...,x_m)$ R_n the n-dimensional space of all points $Y=(y_1,...,y_n)$, R_{m+n} the m+n-dimensional space of all points $(X,Y)=(x_1,...,x_m, y_1,...,y_n)$.
A set (in R_m or R_n or R_{m+n}) which is not *empty, convex closed* and limited we call a C-set. Be S^o, T^o C-sets in R_m corr. R_n. Be $S^o x T^o$ the set of all (X,Y) (in R_{m+n}), where X transverses the whole of S^o and Y the whole of T^o. Be V, W two closed partial sets of $S^o x T^o$. For every X in S^o be the set Q(X) of all Y with (X, Y) in V a C-set, for every Y in T^o be the set P(Y) of all X with (X, Y) in W a C-set. Then the theorem holds: *Under the above assumptions V, W have (at least) a common point.*
Our problem results by setting $S^o=S$, $T^o=T$ and V=set of all $(X,Y)=(x_1,...,x_m, y_1,...,y_n)$ which fulfill (7*), W=set of all $(X,Y)=(x_1,...,x_m, y_1,...,y_n)$ which fulfill (8*). As easily seen, V, W are closed and the sets $S^o=S$, $T^o=T$, Q(X), P(Y) are all simplices, that is C-sets. The common point of those V, W is naturally the solution $(X,Y)=(x_1,...,x_m, y_1,...,y_n)$ we are looking for.

8. In order to prove the above theorem let S^o, T^o V, W be as described before.
Consider V first. For each X of S^o we choose a point $Y^o(X)$ from Q(X) (e.g. the gravity center of this set). It will generally not be possible to choose $Y^o(X)$ as a continuous function of X. Be $\varepsilon>0$, we define

$$w^{\varepsilon}(X,X') = \mathrm{Max}(0, 1-\tfrac{1}{\varepsilon}\,\mathrm{distance}(X,X')) \tag{14}$$

Let now be $Y^{\varepsilon}(X)$ the center of gravity of $Y^o(X')$ with the relative weighting function $w^{\varepsilon}(X, X')$, where X′ transverses the whole of S^o. I.e.: if

$$Y^o(X) = (y_1^o(X),...,y_n^o(X)),\ Y^{\varepsilon}(X) = (y_1^{\varepsilon}(X),...,y_n^{\varepsilon}(X))$$

then

$$y_j^{\varepsilon}(X) = \int_{S^o} w^{\varepsilon}(X,X')\,y_j^o(X')\,dX' \Big/ \int_{S^o} w^{\varepsilon}(X,X')\,dX' \tag{15}$$

We conclude now on a series of properties for $Y^{\varepsilon}(X)$ (valid for all $\varepsilon>0$):

(i) $Y^\varepsilon(X)$ lies in T^o. Proof: $Y^o(X')$ lies in $Q(X')$, therefore in T^o, and because $Y^\varepsilon(X)$ is a gravity center of points $Y^o(X')$ and T^o is convex, $Y^\varepsilon(X)$ lies also in T^o.

(ii) $Y^\varepsilon(X)$ is (in the whole of S^o) a continuous function of X. Proof: It is sufficient to prove it for every $y_j^\varepsilon(X)$. Now, $w^\varepsilon(X,X')$ is everywhere a continuous function of X,X', $\int_{S^o} w^\varepsilon(X,X')\,dX'$ is always >0, and all $y_j^o(X)$ are limited (they are point coordinates of the limited set S^o). From (15) follows that $y_j^\varepsilon(X)$ is continuous.

(iii) For each $\delta>0$ there is a $\varepsilon_o=\varepsilon_o(\delta)>0$, so that for $0<\varepsilon<\varepsilon_o$ every point $(X, Y^\varepsilon(X))$ has from V a distance $<\delta$. Proof: Suppose the opposite. Then, there would exist a $\delta>0$ and a series $\varepsilon_v>0$ with $\lim_{v\to\infty}\varepsilon_v=0$, so that for every $v=1,2,\ldots$ there exists a X_v in S^o for which $(X_v, Y^{\varepsilon_v}(X_v))$ has a fortiori a distance $\geqq\delta$ from V, then $Y^{\varepsilon_v}(X_v)$ has a distance $\geqq\delta/2$ from every $Q(X')$ with a distance $(X_v, X')\leqq\delta/2$. All X_v, $v=1,2,\ldots$, lie in S^o, therefore they have a culmination point X^* in S^o. Therefore there is a partial series of X_v, $v=1,2,\ldots$, converging towards X^*, in which the distance is always $(X_v, X^*)\leqq\delta/2$.

Substituting ε_v, X_v by this partial series we see that one can assume: $\lim X_v=X^*$, distance $(X_v,X^*)\leqq\delta/2$. Therefore we can set for each $v=1,2,\ldots$ $X'=X^*$ and we have in this way always: $Y^{\varepsilon_v}(X_v)$ has a distance $\geqq\delta/2$ from $Q(X^*)$.

$Q(X^*)$ is convex and therefore the set of all points with a distance $<\delta/2$ from $Q(X^*)$ is also convex. Because $Y^{\varepsilon_v}(X_v)$ does not belong to this set and because it is a gravity center of points $Y^o(X')$ with a distance $(X_v, X')\leqq\varepsilon_v$ (while for a distance $(X_v,X')>\varepsilon_v$ it is following (14) $w^{\varepsilon_v}(X_v,X')=0$, do also not all these points belong to the mentioned set. Therefore there exists a $X'=X'_v$ for which the distance $(X_v,X'_v)\leqq\varepsilon_v$ and $Y^o(X'_v)$ has a distance $\geqq\delta/2$ for $Q(X^*)$.

Because $\lim X_v=X^*$, $\lim$distance $(X_v,X'_v)=0$, it is $\lim X'_v=X^*$. All $Y^o(X'_v)$ belong to T^o and therefore they have a culmination point Y^*. It follows that (X^*,Y^*) is a culmination point of $(X'_v,Y^o(X'_v))$ and, because all these belong to V, it belongs also to V. Therefore Y^* is in $Q(X^*)$. Now, each $Y^o(X'_v)$ has a distance $\geqq \delta/2$ from $Q(X^*)$, therefore the culmination point Y^* also. This is a contradiction and the proof is herewith concluded.

(i)-(iii) together mean: For every $\delta>0$ there exists a continuous mapping $Y_\delta(X)$ from S^o on a partial set from T^o, where every point $(X, Y_\delta(X))$ has a distance $<\delta$ from v.
(Put $Y_\delta(X)=Y^\varepsilon(X)$ with $\varepsilon=\varepsilon_o=\varepsilon_o(\delta)$).

9. Interchanging S^o and T^o as well as V and W results now to: For every $\delta>0$ there exists a continuous mapping $X_\delta(Y)$ of T^o on a partial set of S^o, where each point $(X_\delta(Y),Y)$ has a distance $<\delta$ from W.
Setting $f_\delta(X)=X_\delta(Y_\delta(X))$. $f_\delta(X)$ is then a continuous mapping of S^o on a partial set of S^o. Because S^o is a C-set, i.e. topological a Simplex[2]), we can apply the fixed point theorem of L.E.J. Brouwer[3]). $f_\delta(X)$ has a fixed point. I.e. there exists a X^δ in S^o, for which $X^\delta=f_\delta(X^\delta)=X_\delta(Y_\delta(X^\delta))$.
Let $Y^\delta=Y_\delta(X^\delta)$, then we have $X^\delta=X_\delta(Y^\delta)$. Therefore, the point (X^δ,Y^δ) in R_{m+n} has distances $<\delta$ from V as well as from W. V and W have therefore a distance $<2\delta$.
Because this holds for every $\delta>0$ have V, W a distance O. Because V, W as limited and closed must therefore have a common point. This concludes completely the proof of our theorem.

10. We have solved herewith (7*), (8*), from §4 as well as the equivalent problem (*) from §5, and the original question from §3: The solution of (3)-(8′). If x_i, y_j (which in §§7-9 we have called X, Y) are determined, then α, β result from (13) in (**) in §5. In particular $\alpha=\beta$.
As we have already emphasized in §4, there can by all means be several solutions x_i, y_j (i.e. X, Y), we wish now only show that there is only a unique value for a (i.e. for β). Let indeed be X_1, Y_1, α_1, β_1 and X_2, Y_2, α_2, β_2 two solutions. Then (7**), (8**) and (13) result to:

2. Referring to this as well as to the other here applied properties of convex sets s. e.g. B.P. Alexandroff and H. Hopf "Topologie", Vol. I, J. Springer, Berlin 1935, pp. 598-609.
3. See e.g. l.c.[1] p. 480.

$$\alpha_1 = \beta_1 = \Phi(X_1, Y_1) \leqq \Phi(X_1, Y_2)$$

$$\alpha_2 = \beta_2 = \Phi(X_2, Y_2) \geqq \Phi(X_1, Y_2)$$

therefore $\alpha_1 = \beta_1 \leqq \alpha_2 = \beta_2$. Because of symmetry it is also $\alpha_2 = \beta_2 \leqq \alpha_1 = \beta_1$, therefore it is $\alpha_1 = \beta_1 = \alpha_2 = \beta_2$.
We see therefore:
There exists at least one solution X, Y, α, β. For all solution it holds

(13) $\alpha = \beta = \Phi(X, Y)$,

and has for all solutions the same numerical value in other words:
The interest factor and the economy expansion coefficient are equal and uniquely determined by the technically possible processes $P_1, \ldots, P_m$.
Because of (13) it is $\alpha > 0$, but it can be $\alpha \gtreqless 1$. One would expect $\alpha > 1$, but $\alpha < 1$ can obviously not be excluded from our general consideration: The processes $P_1, \ldots, P_m$ can in reality be unproductive.

11. We wish further to characterize α in two independent ways.
Let us consider first an economy situation which is technically possible and expands with a factor α' per time unit. I.e. for the intensities $x'_1, \ldots, x'_m$ holds

(3′) $x'_i \geqq 0$ (5′) $\sum_{i=1}^{m} x'_i > 0$

and

(7″) $\alpha' \sum_{i=1}^{m} a_{ij} x'_i \leqq \sum_{i=1}^{m} b_{ij} x'_i$

We do not at all consider prices. Let x_i, y_j, $\alpha = \beta$ be a solution of our original problem (3)-(8′) in §3. By multiplying (7″) by y_j and the addition sign $\sum_{j=1}^{n}$ we get:

$$\alpha' \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x'_i y_j \leqq \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x'_i y_j$$

that is $\alpha' \leqq \Phi(X', Y)$. Because of (8**) and (13) in (5) it follows:

(15) $\alpha' \leqq \Phi(X', Y) \leqq \Phi(X, Y) = \alpha = \beta$

Let us secondly consider a price system where the interest factor β' does not allow any profit.
I.e. for the price $y'_1, \ldots, y'_n$ it holds:

(4′) $y'_j \geqq 0$ (6′) $\sum_{j=1}^{n} y'_j = 1$

and

(8″) $\beta' \sum_{j=1}^{n} a_{ij} y'_j \geqq \sum_{j=1}^{n} b_{ij} y'_j$

We do not at all consider production intensities. Let be x_i, y_j, $\alpha = \beta$ like above. By multiplying (8″) by x_i and the addition sign $\sum_{i=1}^{m}$ we get:

$$\beta' \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y'_j \geqq \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x_i y'_j$$

that is $\beta' \geqq \Phi(X, Y')$. Because of (7**) and (13) in §5 it follows:

(16) $\beta' \geqq \Phi(X, Y') \geqq \Phi(X, Y) = \alpha = \beta$

These two results may be also formulated as follows:
The greatest expansion factor α' of the whole economy, which is purely technically possible is $\alpha' = \alpha = \beta$. Where prices are not considered.
The lowest interest factor β' which allows a price system without profit is $\beta' = \alpha = \beta$. Where production intensities are not considered.
Let us notice that these characterizations are possible only because we know of the existence of solutions for the original problem, although they do not refer directly to our problem.
Further, the equality of the maximum in the first and the minimum in the second formulation can only be proven because of the existence of these solutions.