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Some unpopular Propositions for a Sraffa Model

by
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I

We assume a linear, profitable and indecomposable technique of single production $[A, \ell]$. The non-singular matrix A, $A \equiv [\alpha_{ij}] \geq 0$, symbolises the square nXn matrix of technical coefficients[1], the element α_{ij} of which represents the quantity of commodity i required to produce one unit of commodity j (as gross product), with $i, j = 1, 2, ..., n$, while the vector ℓ, $\ell \equiv [\ell_j] > 0$ symbolises the 1Xn vector of inputs in direct homogeneous labor, the component ℓ_j of which represents the quantity of labor required to produce one unit of commodity[2] j (as gross product).

As it is known, if we introduce the usual assumptions, the prices of n commodities produced are determined by the following system of equations:

$$p = pA(1 + r) + w\ell \qquad (1)$$

where p is the 1Xn vector of the prices of n commodities produced, w(r) is the *by assumption* uniform nominal wage rate (rate of profit).

The system (1) consisting of n equations has two degrees of freedom, and in order to determine the prices of the commodities, it is necessary, first, to introduce some normalization equation (of prices) of the form:

$$pu = c \qquad (2)$$

where u is a positive or semi-positive nX1 vector, which we shall call the *normalization commodity* (Stamatis (1984), Ch. II. See also Catz (1987), (1991)) and c is a positive constant, and second, the exogenous determination of the nominal wage rate or rate of profit.

1. If all elements of a matrix (vector) A are greater than those of B, we write $A > B$, if they are greater or equal, we write $A \geqq B$; we write $A \geq B$, if $A \geqq B$, and $A \neq B$.
2. As it is known, the fact that $[A, \ell]$ is profitable means that the largest real eigenvalue of A is less than unit.

As it is known, from (1) and (2) the equation of the so-called w–r curve is deduced:

$$w = c/\ell\,[B(0) + B(r)Hr]u = c/\ell\,B(r)u,\ \forall w \in (0, W] \qquad (3)$$

$$r = R \equiv (1 - \lambda)/\lambda \quad \text{for} \quad w = 0 \qquad (3\alpha)$$

where $B(r) \equiv [I - (1 + r)A]^{-1}$, $H \equiv AB(0)$, I the nxn identity matrix, $W \equiv c/\ell\,B(0)u$ and λ (> 0) the largest real eigenvalue of A. The economically significant interval of r is: [0, R], because to this (and only to this) corresponds a positive prices vector p and a semi-positive[3] w (see, e.g. Tucci (1976), Mariolis (1992)).

From (1), (2), (3), (3α) and the Perron-Frobenius Theorems for semi-positive matrices it follows that:

Proposition 1. Because the price of the normalization commodity can be broken down (into wages and profits) as follows [where $\omega \equiv \ell B(0)$]:

$$pu = pHur + w\omega u \;\Rightarrow$$
$$w = (c/L_u) - (K_u/L_u)r \qquad (4)$$

it follows that the magnitude $L_u \equiv \omega u$ expresses the quantity of labour, the magnitude (c/L_u) (= W) the price productivity of labor, the magnitude $K_u \equiv p(r)Hu$ (which in the interval [0, R) is equal to: $w\ell B(r)Hu$ and for r = R to: c/R) the "quantity of capital" and the magnitude $k_u \equiv K_u/L_u$ the price intensity of capital *in that production subsystem* à la Fel'dman ([1928] [1964] 1996) / Sraffa ([1960] 1985): a) Which produces u as its net product, and b) In which, the price of its net product is exogenously given, constant and equal to c. We shall call this subsystem the *normalization subsystem*.
[See: Stamatis (1984), Ch. IV, (1988), Ch. III, Vouyiouklakis/Mariolis (1992)].

Proposition 2. For the slope of the w–r curve it holds (with y′ we symbolise the derivative for a function y(r)):

$$w' = -\frac{pAB(r)u}{\ell B(r)u} = -(f_r/f_w) = -(k_u r)'\,(<0),\ \forall r \in [0,R) \qquad (5)$$

3. Since every row (column) of the matrix: adj[I–(1+R)A] constitues a left (right) eigenvector of A associated with λ, p(r) and w(r) constitute continuous functions of r at the value r=R.

and

$$w' = -(k_u r)' = -k^*(R)\,(<0), \quad \text{for } r = R \tag{5α}$$

where f_r, f_w the partial derivatives for the *multivariable function* (before the introduction of (2) to the model) $pu = f(r, w)$, $r \neq R$, $w \neq 0$, $k_u r$ the profits per unit of labor in the normalization subsystem and $k^* \equiv pAq^* / \ell q^*$ the price intensity of capital in the Sraffian standard system (Sraffa ([1960] 1985), Ch. IV and V), i.e. the system which produces as its gross product the right eigenvector q^* (> 0) of A associated with[4] λ.
[For the proof: Stamatis (1984), Ch. IV, (1988), Ch. III, (1992), Vouyiouklakis/ Mariolis (1992), Mariolis (1993), Ch. V].

Corollary. The w–r curve is a contour line. For a given technique $[A, \ell]$, the form of the w–r curve depends exclusively on the technical production conditions of any selected normalization commodity. Of course, for the form of the w–r curve to be independent of the technical production conditions for certain commodities, the matrix A must be decomposable (see *Part II*).

II

Let us now assume that the linear and profitable technique $[A, \ell]$ can be written in the form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \ell = [\ell_I, \ell_{II}]$$

where $[A_{11}, \ell_I]$, $[A_{12}, A_{22}, \ell_{II}]$ is the technique of the m ($1 \leq m < n$) basic and of the n–m non-basic processes respectively. We also assume that A_{22} is indecomposable and that the following holds: $\lambda_I \gtrless \lambda_{II} \Rightarrow R_I \lessgtr R_{II}$, where $R_I \equiv (1-\lambda_I)/\lambda_I$, $R_{II} \equiv (1-\lambda_{II})/\lambda_{II}$ are the maximum rates of profit of the basic

4. As it is easily proven, the slope of the w-r curve is a continuous function of r at the value r=R:

$$\lim_{r \to R} \frac{w(r)-w(R)}{r-R} = \lim_{r \to R} \frac{c|A| \prod_{v=1}^{n} (R_v - r)}{\ell \,\text{adj}\,[I-(1+R)A]u} \left(\frac{1}{r-R}\right) = \lim_{r \to R} -\frac{pAB(r)u}{\ell B(r)u}$$

where $|A|$ the determinant of A, $R_v \equiv (1-\lambda_v)/\lambda_v$ and λ_v the $v(=1, 2, ..., n)$ eigenvalue of A.

and non-basic processes[5]. Lastly, with $p \equiv [p_I, p_{II}]$ we shall symbolise the vector of prices of m basic (p_I) and n–m non-basic (p_{II}) commodities and with $u \equiv [u_I, u_{II}]^T$ the normalization commodity. Consequently, system (1) and equation (2) take the form:

$$p_I = p_I A_{11}(1 + r) + w\ell_I \quad (6)$$

$$p_{II} = (p_I A_{12} + p_{II} A_{22})(1 + r) + w\ell_{II} \quad (6\alpha)$$

$$p_I u_I + p_{II} u_{II} = c \quad (7)$$

From the Perron-Frobenius Theorems for semi-positive matrices and (6), (6α), (7) it follows that:

Lemma 1. For the left (right) eigenvector $p^* \equiv [p_I^*, p_{II}^*]\left(q^* \equiv [q_I^*, q_{II}^*]^T\right)$ of A associated with its largest real eigenvalue it holds that:

1. When $\lambda_{II} < \lambda_I : p^* > 0,\ q_I^* > 0,\ q_{II}^* = 0$

2. When $\lambda_{II} = \lambda_I : p_I^* = 0,\ p_{II}^* > 0,\ q_I^* > 0\ q_{II}^* = 0$

3. When $\lambda_{II} > \lambda_I : p_I^* = 0,\ p_{II}^* > 0,\ q^* > 0$

[For the proof: Zaghini (1967), Egidi (1975), Vassilakis (1982), Vouyiouklakis/ Mariolis (1992), (1993)].

Lemma 2. If w = 0, then the system (6), (6α), (7) *may have* an economically significant ($p > 0$, $r > 0$) and/or an economically quasi-significant ($p \geq 0$, $r > 0$) solution (we assume that the w–r curve is not linear):

1. $R_I < R_{II}$:

α. $u_{II} = 0 \Rightarrow \{r = R_I, p > 0\}$

β. $u_{II} \geq 0 \Rightarrow \{r = R_I, p > 0\}$ *and* $\{r = R_{II}, p_I = 0, p_{II} > 0\}$

2. $R_I = R_{II}$:

α. $u_{II} = 0 \Rightarrow \left\{r = R_I, p_I > 0, \lim_{r \to R_I} p_{II}(r) = \pm\infty\right\}$

5. As the maxiumum rate of profit of basic (non-basic) processes, we define that value of the rate of profit which emerges for w = 0 (for w = 0 and for prices of basic commodities equal to zero) and to which positive prices correspond for all the basic (non-basic) commodities. Thus, λ_I (λ_{II}) symbolises the largest real eigenvalue of A_{11} (A_{22}). See also Zaghini (1967), p. 259.

β. $u_{II} \geq 0 \Rightarrow \left\{ r = R_I, p_I = 0, p_{II} > 0 \right\}$

3. $R_I > R_{II}$:

α. $u_{II} = 0 \Rightarrow \left\{ r = R_I, p_I > 0, p_{II} \gtreqless 0 \right\}$

β. $u_{II} \geq 0 \Rightarrow \left\{ r = R_{II}, p_I = 0, p_{II} > 0 \right\}$

[For the proof: Vassilakis (1982), Stamatis (1984), Ch. IV, (1988), (1988α), Vouyiouklakis/Mariolis (1992), (1993), Mariolis (1993), Ch. IV, Papachristos/Stamatis (1994)].

Finally, from the **Proposition 1**, **2**, the equations (6), (6α), (7) and the **Lemma 1**, **2** we obtain:

Proposition 3. If, and only if, $r \in [0, \min\{R_I, R_{II}\})$, then a semi-positive rate of profit, a positive nominal wage rate and *positive* prices are *always* (i.e. $R_I \gtreqless R_{II}$) mutually consistent. *Consequently* (see *Lemma 2*), *the interval of* $r(\geq 0)$ *in which* p *and* w *are semi-positive depends on the composition of the normalization commodity*. [For the proof: Vassilakis (1982), Stamatis (1984), Ch. II, IV, (1988), (1992), Vouyiouklakis/Mariolis (1992), (1993), Mariolis (1992), (1993), Ch. IV].

Proposition 4. At an economically quasi-significant value of r, the slope of the w–r curve may be *positive* or *equal to zero* (thus we can say that some decomposable techniques have properties similar to fixed capital or joint production systems):

1. $R_I < R_{II}$, $\ell q^{**} \neq 0$, where q^{**} the right eigenvector of A associated with λ_{II} *and* $u_{II} \geq 0$ (compare with (5α)):

$$w'(R_{II}) = -p(R_{II})Aq^{**} / \ell q^{**} \quad (8)$$

Therefore, at the value $r = R_{II}$, the slope of the w–r curve may be positive (ceteris paribus, *this depends on vector* ℓ).

2. $R_I = R_{II}$ *and* $u_{II} \geq 0$: the price intensity of capital in the Sraffian standard system is equal to zero (see *Lemma 1, case 2*) and thus the slope of the w–r curve is equal to zero (see (5α)).

[For the proof: Vouyiouklakis/Mariolis (1992), Mariolis (1993), Ch. V, (1996)].

Proposition 5. *Generally*, the appearance of prices that tend to infinity or are negative and *the inability to preclude zero prices* are caused by assuming the existance of a uniform rate of profit for the basic and non-basic processes. *In particular*, however, the change of the interval of r in which p and w are semi-positive, the change in the determinant factors of the w–r curve and of the sign of its slope, as well as the conversion of the prices tending to infinity (positive prices) to positive (zero), as a function of the composition of the normalization commodity, show that the w–r curve constitutes *a geometric locus* (*a contour line* – see (4), (5), (5α)) *which expresses the normalization subsystem*.

Proposition 6. Only by means of the *Proposition 5* can we determine and interpret all the "strange" properties of the decomposable techniques and deepen the investigation in the indecomposable ones[6].

The aforementioned properties are the following:

1. Invalidity of the so-called "non-substitution theorem".
2. The determination of the most profitable technique through the outer envelope of the w–r curves and the determination of the most profitable technique through the cost minimization criterion arrive at the same result, if, and only if, $u_{II} \geq 0$.
3. The classification of techniques (with respect to their profitability) depends on the composition of u, because by changing the composition of u, the economically quasi-significant interval of r changes.
4. The famous independence of the profit rate from the production conditions of the non-basic commodities holds if, and only if, $u_{II} = 0$. But then, this independence is a tautology. If $u_{II} = 0$, then the normalization subsystem produces only basic commodities. Therefore the profit rate *cannot* depend on the production conditions of the non-basic commodities. The rule is: for the income distrubition to change as a result of a change in the production conditions of a commodity, this commodity must enter (directly or indirectly) the production of the normalization

6. For example: in the case where only basic commodities are produced, why do the intersection points of the w–r curves, which do not belong to the outer envelope of the w–r curves (i.e. the so-called "false switch points") depend on the composition of the normalization commodity? If the w–r curve was a curve which unambiguously characterised each production technique, then the intersection points of *any* two w–r curves would *always* be independent of the composition of the normalization commodity.

commodity (:necessary condition, because there is the case of the "switch point"). In addition (*Lemma 2* and *Proposition 3*), if, and only if, $u_{II} \geq 0$ holds, then the semi-positiveness of the prices is ensured in all cases.

[For the proof of *Propositions 5,6:* Vassilakis (1982), Stamatis (1984), Ch. IV, (1988), (1990), (1992), Vouyiouklakis/Mariolis (1992), Mariolis (1994), (1996), Sotirchos (1997)].

The previous unpopular propositions identify, link and interpret the unexpected[7] problems associated with the introduction of a price normalization to a Sraffa model of *production prices*. Naturally, these are not problems which pertain to economic reality per se, but to its representation. Therefore, their existence –without prejudicing the neo-Ricardian criticism of the neo-classical theory– illustrates the inner limits of a Sraffa model as cohesive representation of economic reality.

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7. For example, Kurz and Salvadori (1998), p. 416, quite rightly note that an objective property of the economic system under consideration "must be totally independent of the numeraire adopted... On the contrary, the numeraire is chosen by the observer at his or her will and is not related to an objective property of the economic system, apart from the obvious fact that the numeraire must be specified in terms of valuable things (e.g. commodities, labour) that are a part of the economy that is being studied".

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