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The so-called Temporal Single System (TSS) the so-called Standard Simultaneous Methodology (SSM) and the correct definition of labour values*

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1. The Subject

In this article, the so-called Standard Simultaneous Methodology (SSM) and the so-called Temporal Single System (TSS) approach to determining labour values will be presented, compared and evaluated. We shall show that neither SSM constitutes an approach to static states of equilibrium, nor does TSS constitute an approach to dynamic states of non-equilibrium, and that both SSM and TSS simply constitute two different –more or less expedient, effective or correct– definitions and ways of determining labour values (see also Duménil and Lévy (1998), (1999)). We shall present a definition of labour values similar to TSS, in order to show that this too does not constitute an approach to dynamic states of non-equilibrium. Lastly, we shall give the correct definition of labour values. Owing to the fact that, as is known, labour values in joint production systems are positive but not uniquely determined magnitudes (see Stamatis (1983)), in this article we shall be presupposing the existence of single production. Needless to say, we shall also be presupposing that each single production technique used is productive.

2. SSM and TSS

Let ℓ_t, $\ell_t > 0$, $t = 0, 1, 2, 3, \ldots$, the 1xn vector of inputs of direct homogeneous labour per unit of produced commodity and A_t, $A_t \geq 0$, $t = 0, 1, 2, 3, \ldots$, the nxn matrix of the used-up means of production per unit of produced commodity in period t.[1]

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1. The number n can increase with an increasing t. When an increasing t is accompanied by an increasing n, it means that both the kinds of produced commodities and the number of

Needless to say, we presuppose that each column of A_t is positive or semi-positive. The 1xn vector of SSM-labour values is denoted by ω and the 1xn vector of TSS-labour values by λ. Thus for the SSM-labour values ω_{t+1} in period t+1 we get:

$$\omega_{t+1} = \omega_{t+1} A_{t+1} + \ell_{t+1} \quad (1)$$

and for the TSS-labour values λ_{t+1} in period t+1 we get:

$$\lambda_{t+1} = \lambda_t A_t + \ell_t . \quad (2)$$

The sub-indices in (1) and (2) denote the period to which the respective magnitudes refer. Period 0 is the time interval between point 0 in time and point 1 in time, period 1 is the time interval between point 1 in time and point 2 in time, and so on. Thus, period 0 is the 1st period, period 1 is the 2nd period, and so on.

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production processes increase. However, since it is assumed that the technique is a square single production technique, the number of the kinds of produced commodities remains equal to the number of production processes Assuming that in production period t, the number of the produced commodities and the number of the used production processes is equal to 2, and that in production period t+1 the number of the produced commodities and the number of the production processes is equal to 3. Then, if for A_{t+1} and ℓ_{t+1} the following hold

$$A_{t+1} = \begin{bmatrix} (\alpha_{11})_{t+1} & (\alpha_{12})_{t+1} & (\alpha_{13})_{t+1} \\ (\alpha_{21})_{t+1} & (\alpha_{22})_{t+1} & (\alpha_{23})_{t+1} \\ (\alpha_{31})_{t+1} & (\alpha_{32})_{t+1} & (\alpha_{33})_{t+1} \end{bmatrix} (\geq 0)$$

and

$$\ell_{t+1} = \begin{bmatrix} (\ell_1)_{t+1} & (\ell_2)_{t+1} & (\ell_3)_{t+1} \end{bmatrix} (>0)$$

for A_t and ℓ_t we write

$$A_t = \begin{bmatrix} (\alpha_{11})_t & (\alpha_{12})_t & 0_t \\ (\alpha_{21})_t & (\alpha_{22})_t & 0_t \\ 0_t & 0_t & 0_t \end{bmatrix} (\geq 0)$$

and

$$\ell_t = \begin{bmatrix} (\ell_1)_t & (\ell_2)_t & 0_t \end{bmatrix} (\geq 0)$$

For the sake of convenience however, we shall assume that n remains constant.

Since the technique $[A_t, \ell_t]$, used in each production period is by assumption productive, the following holds:

$$(0 <)\, \lambda^{A_t} < 1, \quad \forall t, \tag{3}$$

where λ^{A_t} is the non-repeated maximal eigenvalue of A_t.

By successively replacing (1) itself on the right side of (1), we get:

$$\begin{aligned} \omega_{t+1} = {} & \ell_{t+1} + \\ & + \ell_{t+1} A_{t+1} + \\ & + \ell_{t+1} A_{t+1}^2 + \\ & + \ell_{t+1} A_{t+1}^3 + \\ & \quad \vdots \\ & + \ell_{t+1} A_{t+1}^{k-1} + \\ & + \omega_{t+1} A_{t+1}^{k} . \end{aligned} \tag{1a}$$

Consequently, for a sufficiently large k, we get:

$$\begin{aligned} \omega_{t+1} \cong {} & \ell_{t+1} + \\ & + \ell_{t+1} A_{t+1} + \\ & + \ell_{t+1} A_{t+1}^2 + \\ & + \ell_{t+1} A_{t+1}^3 + \\ & \quad \vdots \\ & + \ell_{t+1} A_{t+1}^{k-1} . \end{aligned} \tag{1b}$$

From (2) we get:

$$\begin{aligned} \lambda_{t+1} = {} & \ell_t + \\ & + \ell_{t-1} A_t + \\ & + \ell_{t-2} A_{t-1} A_t + \\ & + \ell_{t-3} A_{t-2} A_{t-1} A_t + \\ & \quad \vdots \\ & + \ell_0 A_{t-(t-1)} A_{t-(t-2)} A_{t-(t-3)} \cdots A_t + \\ & + \lambda_0 A_0 A_{t-(t-1)} A_{t-(t-2)} A_{t-(t-3)} \cdots A_t . \end{aligned} \tag{2a}$$

Equation (4bis) of Giussani (1998) follows directly from (2a).

As is immediately clear from (2a), λ_{t+1} cannot be calculated, because λ_0 is unknown. In order to calculate λ_{t+1}, λ_0 must be exogenously given. This exogenous and arbitrary setting of λ_0 constitutes the so-called initial condition of the determination of λ_{t+1}.

Giussani (1998) sets λ_0 exogenously by

$$\lambda_0 = \lambda_0 A_0 + \ell_0 .$$

This relation gives

$$\lambda_0 = \ell_0 (I - A)^{-1} .$$

However, because according to SSM the following holds

$$\omega_0 = \omega_0 A_0 + \ell_0 \quad \Rightarrow$$
$$\omega_0 = \ell_0 (I - A)^{-1} ,$$

the following also holds

$$\ell_0 = \omega_0 .$$

Thus, Guissani determines the TSS-labour values λ_0 in period 0, as he explicitly states, as SSM-labour values.

Suppose then that B is a primitive matrix for which the following hold

$$B \geq 0 ,$$
$$(0 <)\ \lambda_m^B < 1 ,$$

and consequently for $t \rightarrow \infty$

$$B^t \rightarrow 0 ,$$

where λ_m^B is the maximal eigenvalue of matrix B. Suppose, also, that

$$B \geqq A_0$$
$$B \geqq A_{t-(t-1)}$$
$$B \geqq A_{t-(t-2)}$$
$$\vdots$$
$$B \geqq A_t . \tag{3a}$$

With respect to the following, we presuppose the validity of (3a). The presupposition (3a) clearly means that the maximal eigenvalues of matrixes A_0,

$A_1, A_2, \ldots$ are all (positive and) smaller than unit. Consequently, (3a) is identical to (3). Then obviously

$$B^{t+1} \geqq A_0 A_{t-(t-1)} A_{t-(t-2)} \cdots A_t .$$

Therefore, when $t \to \infty$, then

$$A_0 A_{t-(t-1)} A_{t-(t-2)} \cdots A_t \to 0 .$$

and consequently, *irrespective of the absolute magnitude of the components of vector* λ_0, the following holds:

$$\lambda_0 A_0 A_{t-(t-1)} A_{t-(t-2)} \cdots A_t \to 0 .$$

Thus for $t \to +\infty$ and for a sufficiently great t, the following holds:

$$\begin{aligned} \lambda_{t+1} \cong\ & \boldsymbol{l}_t + \\ & + \boldsymbol{l}_{t-1} A_t + \\ & + \boldsymbol{l}_{t-2} A_{t-1} A_t + \\ & + \boldsymbol{l}_{t-3} A_{t-2} A_{t-1} A_t + \\ & \quad \vdots \\ & + \boldsymbol{l}_0 A_{t-(t-1)} A_{t-(t-2)} A_{t-(t-3)} \cdots A_t . \end{aligned} \tag{2b}$$

Consequently, according to (2b), for a sufficiently great t, the TSS-labour values λ_{t+1} of period $t+1$ are almost completely independent of the exogenously and arbitrarily set TSS-labour values λ_0 of period 0, i.e. of the initial condition.

3. A Quantitative Comparison of TSS and SSM values

Let us now quantitatively compare the TSS-values λ_{t+1} of period $t+1$ with the SSM-values ω_{t+1} of period $t+1$. The former are determined by (2a)[2]. The latter are determined either by (1a) or by (1b). We shall use relation (1b) to determine the SSM-values ω_{t+1} in period $t+1$, presupposing that k is sufficiently great[3].

2. The case where t is not sufficiently great, and consequently (2b) does not hold, is thus covered.
3. This presupposition is in no way arbitrary. For k represents the number of logical steps, but not of *actual* steps, in the calculation process of ω_{t+1}. Of course, we can consider as many calculating steps as we wish.

It follows directly from the comparison of (2a) and (1b) that the relation between λ_{t+1} and ω_{t+1} depends on the following two factors:

a) The evolution in time of ℓ_t and A_t and
b) The initial condition, i.e. the arbitrarily set λ_0

and that it varies when these two factors vary.

The weight of the second factor, i.e. of the initial condition, is as greater (smaller) as t is smaller (greater). When t is too great and consequently (2b) holds instead of (2a), the influence of the initial condition λ_0 on λ_{t+1} almost vanishes.

Let us accept that the productivity of labour increases constantly. For the sake of convenience, we shall also accept that the productivity of labour increases constantly, because when t increases, ℓ_t and/or A_t decreases, i.e. because the following holds:

$$(0<)\,\ell_t \le \ell_{t-1} \le \ell_{t-2} \le \dots \le \ell_{t-(t-1)} \le \ell_0 \quad \text{or/and}$$
$$(0\le)\,A_t \le A_{t-1} \le A_{t-2} \le \dots \le A_{t-(t-1)} \le A_0 .\text{[4]} \qquad (4)$$

In the case when

(a) the productivity of labour increases according to (4) and
(b) the initial condition is the one set by Giussani, i.e. the condition

$$\lambda_0 = \lambda_0 A_0 + \ell_0 \;(=\omega_0 > 0),$$

then, first of all, as emerges from (1b) and (2a), the following holds

$$\lambda_{t+1} > \omega_{t+1} \quad \Rightarrow$$
$$\lambda_{t+1} - \omega_{t+1} > 0$$

and, secondly, the components of vector $\lambda_{t+1} - \omega_{t+1}$ are as greater (smaller) as the increase of labour productivity according to (4) is greater (smaller).

However, while with an increasing productivity of labour according to (4), the SSM-values ω_{t+1} decrease, it is possible for the TSS-values λ_{t+1} to increase at the same time (see also Duménil and Lévy (1998), *figure 1*).

4. The productivity of labour can of course increase in other ways. For example, it can at times increase even if some components of ℓ increase while other components of ℓ decrease and some elements of A increase while other elements of A decrease.

The following numerical example serves to clarify the above. Suppose that the economy produces only one commodity and uses this commodity as a means of production. Then ℓ_t and A_t are obviously scalars. Suppose also that ℓ_t remains constant, while A_t decreases as t increases, and thus the productivity of labour increases as t increases. Suppose additionally that t=2, $\ell_0 = \ell_1 = \ell_2 = \ell_3 = \ell = 1$, $A_0 = 9/10$, $A_1 = 3/4$, $A_2 = 2/4$, $A_3 = 1/4$ and $\lambda_0 = 0.01$. Then

$$\omega_0 = 10,\ \omega_1 = 4,\ \omega_2 = 2,\ \omega_3 = 4/3 \ \text{ and}$$
$$\lambda_0 = 0.01,\ \lambda_1 = 1.009,\ \lambda_2 = 1.757 \text{ and } \lambda_3 = 1.878\,.$$

Hence, for λ_0 (=0.01) < ω_0 (=10), with an increasing productivity of labour according to (4), and consequently with a decreases ω_t, λ_t increases («productivity paradox»).

But if we set, ceteris paribus, $\lambda_0 = \omega_0 (=10)$, then we get:

$$\lambda_0 = 10,\ \lambda_1 = 10,\ \lambda_2 = 8.5 \text{ and } \lambda_3 = 5.25\,.$$

Consequently, for $\lambda_0 = \omega_0$ and with an increasing productivity of labour according to (4), the TSS-values λ_t decrease as the productivity of labour increases (for the behaviour of $\lambda_{t+1} - \omega_{t+1}$ with respect to t, see also Duménil and Lévy (1998), *Section A*).

Hence, Giussani (1998) correctly maintains that the «paradox», ascertained by Duménil and Lévy (1998), namely that the TSS-values λ_t always increase with an increasing productivity of labour according to (4), does not appear if one presupposes that $\lambda_0 = \omega_0$ (note that this presupposition is arbitrary, as every other relevant presupposition).

The following however still holds: the inverses of the components of the SSM-values ω_t represent the productivities of labour in the production of the corresponding commodities in period t, while the inverses of the components of the TSS-values λ_t are not related to the productivities of labour in the production of the corresponding commodities. Only when ℓ_t and A_t do not vary with t, and hence the productivity of labour remains constant, in which case, as we shall see below, TSS and SSM coincide (this is only true if $\lambda_0 = \omega_0$), do the inverses of the components of the TSS-values λ_t in period t represent the productivities of labour in the production of the corresponding commodities in period t.

4. Is TSS an approach to dynamic non-equilibrium states and SSM an approach to static equilibrium states?

The formulators and advocates of TSS consider it to be an approach to dynamic non-equilibrium states and SSM an approach to static equilibrium states.

But what is the reality?

Indeed, (2a) gives the impression that TSS is an approach to dynamic states. However, this impression is misleading. Because in actual fact (2a) implies nothing more than what (2) implies. And what does (2) imply? Quite simply, (2) implies the following: The λ^i_{t+1} value of commodity i in period t+1 is the sum of:

a) the value $\lambda_t A^j_t$ (where A^j_t is the jth column of A_t and i=j) of the means of production that was used up in the previous production period t for the production of one unit of commodity i, reckoned in values λ_t of the previous period t, and
b) the direct labour ℓ^j_t (where ℓ^j_t is the jth component of ℓ_t and i=j) that was expended in the production of one unit of commodity i in the previous period t.

That is, (2) implies that the commodities that enter the market in period t+1 were produced in the previous period t and consequently their labour value in period t+1, in which they enter the market, is the sum of the *living and dead labour* that was expended for their production in the immediately previous period t, in which they were produced.

The question as to whether this definition is correct or expedient will be examined later. What is certain however is that this definition in no way implies that TSS is an approach to dynamic non-equilibrium states. The fact that TSS is not an approach to dynamic non-equilibrium states is obvious, as TSS does not contain any information on the produced and demanded quantities of commodities or their evolution in time (see also Stamatis (1998)).

Nor is TSS an approach of dynamic states, because the «dynamic» property of TSS, when it exists and in the way it exists, is not a property of TSS itself, but rather depends on the evolution of ℓ_t and A_t in time. Only when ℓ_t and/or A_t vary in time, does TSS appear to be –without actually being– an approach to dynamic states.

The existence of a dynamic state is the result of the fact that ℓ_t and/or A_t

vary in time. The variation of ℓ_t and/or A_t in time does not imply that TSS, in contrast with SSM which is supposedly an approach to static states, is an approach to dynamic states, because the dynamic state, which arises when ℓ_t and/or A_t vary, is described not only by TSS but also, albeit in a different way, by SSM.

Lastly, when ℓ_t and A_t remain constant in time, then, for $\lambda_0 = \omega_0$, SSM and TSS coincide. For $\ell_t = \ell$ = constant and $A_t = A$ = constant, we get the following for the SSM-values in period t:

$$\omega_t = \omega_t A + \ell_t \;\Rightarrow$$
$$\omega_t = \ell(I-A)^{-1} = \ell(I + A + A^2 + A^3 + \ldots), \;\; \forall t, \qquad (5)$$

and consequently

$$\omega_0 = \omega_1 = \omega_2 = \omega_3 = \ldots = \ell(I + A + A^2 + A^3 + \ldots). \qquad (6)$$

For the λ_t TSS-values in period t we get:

$$\lambda_t = \lambda_{t-1} A_{t-1} + \ell_{t-1} \;\Rightarrow$$
$$\lambda_t = \lambda_{t-1} A + \ell, \;\; \forall t. \qquad (7)$$

Because (7) holds for every t, when $\lambda_0 = \omega_0$, then

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \ldots \qquad (8)$$

Taking into account (8), from (7) we get:

$$\lambda_t = \lambda_t A + \ell, \;\; \forall i, \;\Rightarrow$$
$$\lambda_t = \ell(I-A)^{-1} = \ell(I + A + A^2 + A^3 + \ldots), \;\; \forall t \qquad (9)$$

and consequently

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \ldots = \ell(I + A + A^2 + A^3 + \ldots). \qquad (10)$$

From (5) and (9) or from (6) and (10) we get:

$$\omega_t = \lambda_t, \;\; \forall t. \qquad (11)$$

If TSS was an approach to dynamic states and SSM an approach to static states, as TSS supporters maintain, then in the case of $\lambda_0 = \omega_0$, in which ℓ_t and A_t remain constant, and therefore, as we showed, TSS-values and SSM-values coincide, either SSM *too* should constitute an approach to dynamic states or TSS should constitute an approach to static states – which according to TSS supporters are both impossible. And indeed, neither is the case because, contrary to what TSS supporters maintain, TSS is not an approach to dynamic states nor SSM an approach to static states – and this is true irrespective of whether ℓ_t and A_t vary in time or not[5].

The solution to this apparent enigma is the following: TSS and SSM do not constitute different approaches to states – dynamic or static. They are simply different definitions of labour values. The question as to whether they are correct or not, suitable to their purpose or not, is not the issue at hand. The only difference between them is the following: according to SSM, the commodities that enter the market in period t+1 were produced in that same period t+1. Therefore, according to SSM, the quantities of living and dead labour expended for the production of commodities in period t+1 determine their values in that same period t+1, since according to SSM the commodities by definition enter the market in the same period that they were produced. According to TSS, the commodities, which enter the market in period t+1, were produced in the previous period t. Thus, according to TSS, the quantities of living and dead labour expended for the production of commodities in period t determine their values in the next period t+1 because, according to TSS, the commodities by definition enter the market in the period immediately following the period in which they were produced.

Below, we shall examine a common feature of the labour value definition according to TSS and the labour value definition according to SSM, which (common feature) is more important than their aforementioned difference.

5. NSA or PDA

In order to make it clearer that SSM and TSS are simply different definitions of the labour values of commodities, another definition of labour values will be presented here. Suppose μ is the 1xn vector of labour values, which is defined for the period t+1 as follows:

$$\mu_{t+1} = \mu_t A_{t+1} + \ell_{t+1}. \tag{12}$$

According to (12), the value μ^i_{t+1} of every commodity i in period t+1 is equal to the sum of living labour ℓ_{t+1} expended in the same period t+1 and the value $\mu_t A^j_{t+1}$ of the means of production A^j_{t+1}, i = j, that were used up in the same period t+1 for the production of the commodity i, but reckoned at the labour values μ_t of the previous period t.

5. A comparison of SSM and TSS for a constant productivity of labour is presented in Stamatis 1998, not only with respect to labour values but also production prices.

From (12) we get:

$$\begin{aligned}\mu_{t+1} = {} & \ell_{t+1} + \\ & + \ell_t A_{t+1} + \\ & + \ell_{t-1} A_t A_{t+1} + \\ & + \ell_{t-2} A_{t-1} A_t A_{t+1} + \\ & \vdots \\ & + \ell_{t-(t-1)} A_{t-(t-2)} A_{t-(t-3)} \cdots A_t A_{t+1} + \\ & + \mu_0 A_{t-(t-1)} A_{t-(t-2)} \cdots A_t A_{t+1} . \end{aligned} \tag{13}$$

Under the presumption (3a) and for a sufficiently great t, the following obviously holds:

$$\begin{aligned}\mu_{t+1} \cong {} & \ell_{t+1} + \\ & + \ell_t A_{t+1} + \\ & + \ell_{t-1} A_t A_{t+1} + \\ & + \ell_{t-2} A_{t-1} A_t A_{t+1} + \\ & \vdots \\ & + \ell_{t-(t-1)} A_{t-(t-2)} A_{t-(t-3)} \cdots A_t A_{t+1} . \end{aligned} \tag{13a}$$

We call the above definition of labour values Non-Simultaneous Approach (NSA) or Pseudodynamic Approach (PDA).

This definition of labour values does not constitute an approach to dynamic states either, as TSS supporters would surely be inclined to assert. The misleading impression of an approach to dynamic states is created by the variation of ℓ_t and A_t in time. When ℓ_t and A_t do not vary in time, then for $\mu_0 = \lambda_0 = \omega_0$ (13) has the following form:

$$\mu_{t+1} = \ell(I + A + A^2 + A^3 + \ldots) \tag{13b}$$

and hence PDA (or NSA), SSM and TSS coincide.

6. Comparison between SSM, TSS and NSA (or PDA) as different definitions of labour values

Both SSM and TSS, as well as NSA or PDA, simply constitute different definitions of labour values and correspondingly different ways of calculating labour values. They are not an approach to either static or dynamic states, and they are compatible with both an approach to static states and an approach to dynamic states.

The SSM-value, i.e. the value defined according to Marx ω^i_{t+1}, $\omega^i_{t+1} = \omega^i_{t+1} A^j_{t+1} + \ell^j_{t+1}$, i = j, of each commodity i in period t+1 is not equal to the social average quantity of living and dead labour that the production of this commodity *actually* cost in period t+1, but rather is equal to the social average quantity of living and dead labour that the production of commodity i *would cost if the means of production* A^j_{t+1} *used up in its production had been produced in the period in which they were used up, i.e. in period t+1.* The means of production A^j_{t+1} were however not produced in period t+1, in which they were used up and in which commodity i was produced, but in previous periods.

If the SSM-value ω^i_{t+1} of commodity i in period t+1 had been defined so as to represent the quantity of social average living and dead labour that the production of commodity i *actually* cost in period t+1, then the production means A^j_{t+1} that were used up in the production of commodity i in period t+1 should have been reckoned not at the values ω_{t+1} that apply in period t+1, in which they were used up, but at the values that were applicable in the periods in which these means of production were produced. That is, the components of vector ω, with which the means of production A_{t+1} in period t+1 should be multiplied, ought to have time indices smaller than t+1. If one coefficient of ω had a time index t+1, then this would mean that the corresponding mean of production was produced and used up in the same period t+1 and consequently that it is an intermediate output.

So SSM treats the used up means of production *as if* they were all intermediate outputs, since it reckons the used up means of production A_{t+1} in period t+1 at the values ω_{t+1} of the same period t+1.

TSS does exactly the same thing. TSS also treats the used up means of production *as if* they were all intermediate outputs, since it reckons the used up means of production A_{t+1} in period t+1 at the values λ_{t+1} of the same period. It is of no consequence that, according to TSS, the value λ_{t+1} of the used up means of production A_{t+1} in period t+1 and the living labour ℓ_{t+1} that was expended in period t+1 do not determine, as they do in SSM, the values λ_{t+1} of the commodities in the same period t+1, but rather the values λ_{t+2} of the commodities in the immediately following period t+2. For this difference simply arises from the fact that, while according to SSM the commodities enter the market in the same period in which they were produced, according to TSS the commodities enter the market in the period immediately following the period in which they were produced.

What is the difference between NSA or PDA on the one hand and SSM and TSS on the other? According to NSA or PDA, as in the case of SSM, commodities enter the market in the same period that they were produced and not, as in the case of TSS, in a period after the period in which they were produced. However, while both SSM and TSS reckon the used-up means of production A_{t+1} of period t+1 at the values of the same period t+1, NSA or PDA reckon the used-up means of production A_{t+1} of period t+1 at the values of the immediately previous period t. So, while SSM and TSS implicitly and arbitrarily presuppose that the used-up means of production A_{t+1} of period t+1 were produced in the same period t+1, NSA or PDA implicitly and equally arbitrarily presuppose that the used-up means of production A_{t+1} of period t+1 were all produced in the immediately previous period t. Thus, both SSM and TSS, as well as NSA or PDA implicitly and arbitrarily presuppose that none of the means of production A_{t+1} of period t+1 were produced in a period prior to period t. SSM and TSS in addition implicitly and arbitrarily presuppose that none of the used-up means of production A_{t+1} of period t+1 were produced in period t, but that all of them were produced in period t+1 (in contrast with NSA or PDA, according to which all were produced in period t).

7. Comparison between SSM, TSS and NSA or PDA with respect to their expediency

It has already been shown that SSM, TSS and NSA or PDA are simply different modes of definition and calculation of values. It has also been shown that all three are more or less arbitrary. Below therefore, we shall compare the three aforementioned definitions of labour values with respect to their expediency and «convenience» in the calculation of values.

SSM has the following advantage: The inverse of the SSM-values ω_{t+1} in period t+1 represent, as is known, the productivities of labour in the production of the corresponding commodities in period t+1 – under the presupposition that the used-up means of production in period t+1 were produced in the same period. This holds neither for the TSS-values λ nor for the NSA or PDA-values μ. However, it should be noted that for $\omega_0 = \lambda_0 = \mu_0$ and/or for a sufficiently great t, both the TSS-values λ, and the NSA or PDA-values μ decrease with increasing productivity of labour.

However, SSM has a disadvantage compared to both TSS and NSA or PDA. Not only SSM, but also TSS and NSA or PDA implicitly and arbitrarily presuppose that the means of production A_{t+1} used-up in period t+1 are reproduced in the same period. But this presupposition is not always valid. Some of the used-up means of production are not reproduced in the same period of production in which they were used-up. When this is the case, the production system ceases to be square, since certain commodities and specifically the used-up but non-reproduced means of production are found in inputs but not in outputs. The corresponding system of determination of equations of SSM-values ω contains unknown variables, the number of which is at least one unit greater than that of the independent equations[6]. Consequently, the SSM-values ω cannot be uniquely determined.

Consider the following example:

$$A_{t-1} = \begin{bmatrix} (\alpha_{11})_{t-1} & (\alpha_{12})_{t-1} \\ (\alpha_{21})_{t-1} & (\alpha_{22})_{t-2} \end{bmatrix} (>0)$$

and

$$\ell_{t-1} = [(\ell_1)_{t-1},\ (\ell_2)_{t-1}]\ (>0).$$

On the basis of these data, it is clear that not only the SSM-values ω_{t-1} $(=\omega_{t-1}A_{t-1}+\ell_{t-1})$, but also the TSS-values λ_t $(=\lambda_{t-1}A_{t-1}+\ell_{t-1})$ and the NSA or PDA-values μ_{t-1} $(=\mu_{t-2}A_{t-1}+\ell_{t-1})$ can be uniquely determined. Suppose, moreover, that in the next period t, the production technique changes so that, inter alia, commodity 1, although still used as an input is no longer produced by the production system. Hence we have

$$A_t = \begin{bmatrix} 0_t & (\alpha_{12})_t \\ 0_t & (\alpha_{22})_t \end{bmatrix} \text{ with } (\alpha_{12})_t, (\alpha_{22})_t > 0$$

and

$$\ell_t = [0_t,\ (\ell_2)_t] \text{ with } (\ell_2)_t > 0.$$

It is clear that the SSM-values ω_t $(= \omega_t A_t + \ell_t)$, i.e. the value $(\omega_2)_t$ $[= (\omega_1)_t(\alpha_{12})_t$

6. We presuppose that in period 0 the production system is square and hence that in the same period the system for determining SSM-values includes as many unknowns as independent equations.

$+ (\omega_2)_t(\alpha_{22})_t + (\ell_2)_t]$ of the only produced commodity 2, cannot be uniquely determined, since the corresponding system of equations which determines the SSM-values ω_t consists of one equation with two unknown variables.
On the contrary, the TSS-values λ_{t+1} $(= \lambda_t A_t + \ell_t)$, i.e. here, the value $(\lambda_2)_{t+1}$ $[= (\lambda_1)_t(\alpha_{12})_t + (\lambda_2)_t(\alpha_{22})_t + (\ell_2)_t]$ of the only produced commodity 2, can be uniquely determined, because the TSS-value $(\lambda_1)_t$ is known from the previous production period t. The same holds for the NSA or PDA-values μ_t $(= \mu_{t-1}A_t + \ell_t)$, i.e. for the value $(\mu_2)_t$ $[= (\mu_1)_{t-1}(\alpha_{12})_t + (\mu_2)_{t-1}(\alpha_{22})_t + (\ell_2)_t]$ of the only produced commodity 2. The magnitude $(\mu_2)_t$ can be determined, because $(\mu_1)_{t-1}$ and $(\mu_2)_{t-1}$ are known from period t–1.

8. The actual values ξ (the ξ-Approach)

We noted previously how the labour values should be defined in order to represent the quantities of living and dead labour that were *actually* expended for the production of one unit of each commodity (see also Stamatis (1977), Chapter V. 6). In this section, we shall present the mathematical formula for this definition of labour values.

Let C_t, $C_t \geq 0$ be the matrix that derives from the matrix of intermediate outputs and inputs in period t per unit of produced commodity, when we replace all the elements of its principal diagonal with zero and D_t, $D_t \geq 0$ the nxn matrix of the means of production used up per unit of produced commodity in period t. The used-up means of production of period t existed at the beginning of period t, consequently they were not produced in period t, unlike the intermediate outputs and inputs of period t, but had been produced in previous periods. Let also ℓ_t, $\ell_t > 0$ be the 1xn vector of the inputs of living labour per unit of produced commodity in period t. Then, for the 1xn vector of labour values ξ_t in period t, the following holds:

$$\xi_t = \xi_t C_t + \bar{\xi} D_t + \ell_t \Rightarrow$$

$$\xi_t (I - C_t) = \bar{\xi} D_t + \ell_t$$

Under the self-evident presupposition that

$$(I - C_t)^{-1} \geq 0,$$

for ξ_t we get

$$\xi_t = (\bar{\xi} D_t + \ell_t)(I - C_t)^{-1} =$$

$$= (\bar{\xi} D_t + \boldsymbol{\ell}_t)(I + C_t + C_t^2 + C_t^3 + ...), \tag{14}$$

where $\bar{\xi}$ is the 1xn vector of the dated labour values of the used-up means of production in period t, i.e. of the labour values that the used-up means of production of period t had in the periods in which they were produced[7].

Relation (14) means that the labour value $\xi_t^{(i)}$ of one unit of commodity i in period t is the sum of:

(a) the direct dead labour $\bar{\xi} D_t^{(i)}$ and the direct living labour $\boldsymbol{\ell}_t^{(i)}$ that was required for the production of one unit of commodity i in period t, and

(b) the direct dead and direct living labour $(\bar{\xi} D_t^{(i)} + \boldsymbol{\ell}_t^{(i)})[C_t^{(i)} + (C_t^2)^{(i)} + (C_t^3)^{(i)} + + ...]$, which was required in period t for the production of the necessary *direct* $C_t^{(i)}$ and *indirect* $(C_t^2)^{(i)} + (C_t^3)^{(i)} + ...$ intermediate inputs for the production of one unit of commodity i, or –in order words– of the *direct* $(\bar{\xi} D_t^{(i)} + \boldsymbol{\ell}_t^{(i)}) C_t^{(i)}$ and *indirect* $(\bar{\xi} D_t^{(i)} + \boldsymbol{\ell}_t^{(i)})[(C_t^2)^{(i)} + (C_t^3)^{(i)} + ...]$ living and dead labour that was required in period t for the production of the necessary intermediate inputs $C_t^{(i)}$ for the production of one unit of commodity i.

When $C_t = 0$, then quite clearly, as emerges from (14), the following holds:

$$\xi_t = \bar{\xi} D_t + \boldsymbol{\ell}_t. \tag{14a}$$

According to (14a), the value $\xi_t^{(i)}$ of one unit of commodity i in period t is equal to the sum of *direct* dead labour $\bar{\xi} D_t^{(i)}$ and *direct* living labour $\boldsymbol{\ell}_t^{(i)}$ that was required for the production of one unit of commodity i in period t.

If the nX1 vector $\bar{X}_t$, $\bar{X}_t > 0$ denotes the gross production (=gross product plus intermediate outputs) of period t, then the following holds

7. It is possible for two or more or all production processes to use one or more identical means of production, which however were produced in different production periods and therefore –though identical– have different values. In such a case, the place of vector $\bar{\xi} D_t$ in (14) is taken by vector $(\bar{\xi}^{(1)} D_t^{(1)}, \bar{\xi}^{(2)} D_t^{(2)}, ..., \bar{\xi}^{(n)} D_t^{(n)})$, where $D_t^{(1)}, D_t^{(2)}, ..., D_t^{(n)}$ the first, the second,..., the n^{th} column of D_t and $\bar{\xi}^{(1)}, \bar{\xi}^{(2)}, ..., \bar{\xi}^{(n)}$ the vectors of dated values corresponding to the first, the second,..., the n^{th} production process.

$$\bar{X}_t = X_t + C_t\bar{X}_t, \tag{15}$$

where $C_t\bar{X}_t$ the total intermediate inputs and outputs of period t and X_t, $X_t \geq 0$, the nx1 vector of the gross product of period t.

From (15) we get:

$$X_t = (I - C_t)\bar{X}_t \tag{16}$$

and

$$\bar{X}_t = (I - C_t)^{-1}X_t. \tag{17}$$

For the net product Y_t, the following holds

$$Y_t = X_t - D_t\bar{X}_t \tag{18}$$

and, taking into account (17),

$$Y_t = X_t - D_t(I - C_t)^{-1}X_t \Rightarrow \tag{19}$$

$$Y_t = [I - D_t(I - C_t)^{-1}]X_t. \tag{20}$$

When the used-up means of production $D_t\bar{X}_t$ $[= D_t(I - C_t)^{-1}X_t]$ of period t are fully re-produced in period t, then the following holds:

$$Y_t \geq 0.$$

On the contrary, when in period t, the used-up means of production $D_t\bar{X}_t$ are not fully reproduced *and are therefore not part of the gross product X_t of period t*, then, as emerges from (18), the net product Y_t of period t contains –apart from positive or positive and zero quantities– also negative quantities of commodities. These negative quantities of commodities included in Y_t are the used-up and non-reproduced quantities of means of production in period t.

When are ω-values equal to ξ-values? To enable a comparison of SSM and the ξ-Approach, we must clearly presuppose that:

(a) $C_t = 0, \ \forall t, \ t \neq 0,$

(b) $D_t = A_t, \ \forall t, \ t \neq 0$

and

(c) $C_t \equiv D_t = A_t, \ t = 0.$

It clearly follows from presupposition (c) that

(d) $\omega_0 = \xi_0.$

Under the above presuppositions (a), (b), (c) and consequently (d), ω-values are equal to ξ-values only when the productivity of labour does not vary and consequently both ω-values and ξ-values remain constant.

Proof:

For the ω_t values the following holds

$$\omega_t = \omega_t A_t + \ell_t = \ell_t (I - A_t)^{-1}. \tag{21}$$

Under the presuppositions (a), (b), (c) and (d), for the ξ_t values we get from (14):

$$\xi_t = (\bar{\xi} A_t + \ell_t). \tag{22}$$

From (21) and (22) we get,

$$\omega_t = \xi_t,$$

only when $\xi_t = \bar{\xi}$, i.e. when the productivity of labour remains constant. For under presuppositions (a), (b), (c) and (d) and $\xi_t = \bar{\xi}$ from (22) we get

$$\xi_t = \xi_t A_t + \ell_t = \ell_t (I - A_t)^{-1}. \tag{22a}$$

From (21) and (22a) it emerges that[8]

$$\omega_t = \xi_t. \tag{23}$$

Under the same presuppositions (a) to (d), when the productivity of labour increases and consequently

$$\xi_t \le \bar{\xi},$$

then, as results from (21) and (22),

$$\xi_t \ge \omega_t. \tag{24}$$

and when the productivity of labour decreases and consequently

$$\xi_t \ge \bar{\xi},$$

8. We know from previously that when $\lambda_0 = \mu_0 = \omega_0$ and the productivity of labour remains constant, we have

$$\lambda_t = \mu_t = \omega_t, \quad \forall t.$$

Consequently, under the above presuppositions (a), (b), (c) and (d) and for a constant productivity of labour, the following also applies

$$\xi_t = \lambda_t = \mu_t = \omega_t, \quad \forall t.$$

then, as results from (21) and (22),

$$\xi_t \leq \omega_t . \tag{25}$$

The above implies that –under the presuppositions (a), (b), (c) and (d)– for values $\omega_t U_t$ and $\xi_t U_t$ of every basket of commodities U_t, $U_t \geq 0$ that was produced in period t, when the productivity of labour remains constant, the following holds

$$\omega_t U_t = \xi_t U_t ,$$

and when the productivity of labour increases, the following holds

$$\omega_t U_t \leq \xi_t U_t ,$$

and when the productivity of labour decreases

$$\omega_t U_t \geq \xi_t U_t .$$

Thus, under the above presuppositions (a), (b), (c) and (d), for values $\omega_t Y_t$ and $\xi_t Y_t$ of the net product Y_t, when the productivity of labour remains constant the following holds

$$\omega_t Y_t = \xi_t Y_t ,$$

and when the productivity of labour increases, the following holds

$$\omega_t Y_t \leq \xi_t Y_t ,$$

and when the productivity of labour decreases

$$\omega_t Y_t \geq \xi_t Y_t .$$

The SSM-Approach does not take into account the intermediate outputs and inputs. That is, it implicitly presupposes that $C_t = 0$. Consequently, according to SSM, for the net product Y_t the following holds:

$$Y_t = X_t - A_t X_t = (I - A_t) X_t . \tag{26}$$

Thus, according to SSM, it follows from (21) and (26) that:

$$\begin{aligned} \omega_t Y_t &= (\omega_t A_t + \ell_t)(I - A_t) X_t = \\ &= \ell_t (I - A_t)^{-1} (I - A_t) X_t = \\ &= \ell_t X_t , \end{aligned} \tag{27}$$

i.e. that the value $\omega_t Y_t$ of the net product Y_t is equal to the *living* labour $\ell_t X_t$

that is required for the production of the corresponding *gross* product X_t.

According to the ξ-Approach and taking into account (14) and (20), for the value $\xi_t Y_t$ of the net product Y_t the following holds:

$$\xi_t Y_t = (\bar{\xi} D_t + \ell_t)(I - C_t)^{-1}[I - D_t (I - C_t)^{-1}]X_t. \qquad (28)$$

Under the presuppositions (a), (b), (c) and (d), which ensure the comparability of SSM and the ξ-Approach, it follows from (28) that:

$$\xi_t Y_t = (\bar{\xi} A_t + \ell_t)(I - A_t) X_t. \qquad (29)$$

When the productivity of labour remains constant, and consequently $\xi_t = \bar{\xi}$, then, as results from (27) and (29), the value $\xi_t Y_t$ of the net product Y_t is –under the presuppositions (a), (b), (c) and (d), which are entailed by (29)– equal to the *living* labour $\ell_t X_t$ that was required for the production of the corresponding *gross* product X_t.

When the productivity of labour increases, and consequently $\xi_t \le \bar{\xi}$, then, as results from (27) and (29), the value $\xi_t Y_t$ of the net product Y_t is –under the presuppositions (a), (b), (c) and (d), which are entailed by (29)– greater than the *living* labour $\ell_t X_t$ that was required for the production of the corresponding *gross* product X_t.

When the productivity of labour decreases, and consequently $\xi_t \ge \bar{\xi}$, then, as results from (27) and (29), the value $\xi_t Y_t$ of the net product Y_t is –under the presuppositions (a), (b), (c), and (d), which are entailed by (29)– smaller than the *living* labour $\ell_t X_t$ that was required for the production of the corresponding *gross* product X_t.

Due to the fact that the *net* product of every sector of a single production system is an inhomogeneous magnitude, consisting of different commodities, which as a rule additionally contains negative quantities of commodities, while the *gross* product of every sector of such a system is a homogeneous magnitude, consisting of only one commodity, the productivity of labour of a production sector is defined as the ratio of the *gross* product of this sector to the *living* and *dead* labour necessary for its production. The inverses of the TSS-values and the inverses of the NSA-values or the PDA-values are not the productivities of labour in the corresponding sectors, as defined above.

Only the inverses of the ω-values and the ξ-values are equal to the productivities of labour, as defined above. However, there is the following difference between ω-values and ξ-values: the ω-values presuppose that the used-up means of production $A_t X_t$ were produced in period t, while the ξ-values presuppose that the used-up means of production $D_t \bar{X}_t$ were produced in previous periods. Thus, when –under the presuppositions (a), (b), (c) and (d)– the productivity of labour remains constant and consequently $\omega_t = \xi_t = \bar{\xi}$, then the ω-productivities of labour are equal to the ξ-productivities of labour. When –under the presuppositions (a), (b), (c) and (d)– the productivity of labour increases and consequently $\omega_t \leq \xi_t \leq \bar{\xi}$, then the ξ-productivities of labour are smaller than (at least one of them is smaller) or equal to the ω-productivities of labour. Lastly, when –under the presuppositions (a), (b), (c) and (d)– the productivity of labour decreases and consequently $\omega_t \geq \xi_t \geq \bar{\xi}$, then the ξ-productivities of labour are greater than (at least one of them is greater) or equal to the ω-productivities of labour.

It emerges from the above that the ξ-approach is superior to SSM, because it reckons the used-up means of production of period t not, like SSM, at their values in period t, in which they were used-up, but rather at their values in the previous periods, in which they were produced.

The ξ-Approach is also superior to SSM for an additional reason: It directly follows from (14) that, when the production system is square in the initial period ($t = 0$) and becomes rectangular in any subsequent period owing to the fact that one or more used-up means of production ceases to be reproduced, then the ξ-values (and the ξ-productivities) can be calculated, while the ω-values (and the ω-productivities) cannot be determined. This advantage of the ξ-Approach applies also to TSS and NSA or PDA, as shown above. However, the inverses of the values defined by TSS and NSA or PDA, have nothing to do with the productivities of labour, while the inverse of each ξ-value of a commodity is exactly equal to the ratio of the gross product of that commodity to the living and dead labour that was *actually* required for the production of that gross product.

Therefore the ξ-Approach is superior to SSM, TSS and, NSA or PDA in all respects.

However, for the reasons set out above, the ξ-Approach is solely and simply a definition of values –albeit the most correct– and does not constitute, as TSS supporters would perhaps be inclined to maintain, an approach to dynamic non-equilibrium states.

9. Conclusion

Not only SSM and TSS but also NSA or PDA and, lastly, the ξ-Approach, simply constitute definitions, whether correct or not, and corresponding ways of value calculation. None of these definitions and ways of value calculation constitute –nor could any other definition and way of value calculation be able to constitute– an approach to static states of equilibrium or dynamic states of non-equilibrium. However, the ξ-Approach, as a definition of values, is superior to all other definitions.

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