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Vertical Integration and Division of Labor

by

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Introduction and Statement of the Problem

This paper addresses the following two problems in the setting of a closed model:

(P1) Vertical Integration: what determines the number of integrated firms, i.e., those who make their own intermediate goods, as opposed to specialist firms, who buy the intermediate goods they use?

(P2) Production Roundaboutness: what determines the kind of intermediate goods used in the production of final goods?

The main point of the paper in that (P1), (P2) can be usefully seen as two aspects of the same problem, namely that of the equilibrium degree of division of labor; and that they can both be solved at one stroke by the application of the same principle, namely the balance of the two opposing forces generated by increasing returns to scale and strategic behavior, respectively.

This point of view was first taken by Marx (1867): production roundaboutness is identified with the degree of division of labor within firms, while the degree of vertical disintegration is seen as the degree of division of labor among firms. A. Smith (1776) proposed market size as the crucial variable that limits division of labor within firms, while Young (1928) and Stigler (1951) suggested that market size also limits division of labor among firms. The propositions that have been proposed by the last three authors, and whose derivation is the objective of this paper, are the following:

(T1) Increases in market size increase the degree of vertical disintegration.

(T2) Increases in market size increase the degree of production roundaboutness.

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The modern literature on vertical integration is based on the work of Coase (1937), Stigler (1951) and Williamson (1985). The main idea is that transactions can be organized either through markets or within firms: there are costs involved in either case, and agent choose their degree of vertical (and horizontal) integration so as to minimize these costs.

In the case of vertical integration, the nature of these costs is the following: A firm that becomes more integrated, i.e., that makes more of its inputs, avoids costs associated with nonprice-taking behavior of suppliers and uncertainty with regard to the price, availability and quality of those inputs. On the other hand, though, it has to bear alone the fixed costs of production inputs, which was hitherto shared with other input buyers, and it has to pay more in order to maintain the same degree of control over its employees. Models that attempt to model the vertical intergration decision along these lines, though, ignore general equilibrium effects in many respects, and typically they fail to generate (T1) and (T2). To begin with, the behavior of primary factor suppliers and final consumers is exogenous and summarized by cost and demand functions, respectively. Secondly, the initial situation is supposed to be one of vertical disintegration: the numbers of upstream and downstream producers are exogenous. Integration can take place in two ways: either some firms produce both inputs and outputs and drive disintegrated firms out of the market, or upstream and downstream firms merge. Firms compare their pay-offs in the integrated and disintegrated equilibria and act accordingly. These are several problems with this approach. First, the outcome (either complete integration or complete disintegration) is almost never observed in practice. Second, models that consider integration via competition do not explain what happens to the defeated firms' owners. Thirdly, models that consider integration via merger do not explain why merger has to be only vertical and not horizontal, too, and do not have a theory of mergers either, i.e., they do not consider the incentives for coalition formation. Models that follow this approach can be found in Blair and Kaserman (1983, Chs. 3, 4), Perry and Groff (1983), and Porter and Spence (1977).

The uncertainty models of Arrow (1975), Green (1974), and Carlton (1979) also display at least one of these undesirable features. Perry and Groff attempted to generate an equilibrium in which upstream, downstream and integrated firms coexist, by assuming that agents differ in their ability to produce the upstream and downstream goods. Obviously, agents specialize in what they do best, with integrated firms being those that are good at neither end of the product spectrum. It is not clear, though, why agents differ in their ability to produce upstream or downstream in an otherwise uncertaintly-free world with perfect information.

No modern literature on production roundaboutness seems to be available; we study it here both because it helps to determine the degree of vertical integration and because the study of technical progress consists partly in explaining how, and how efficiently, agents exploit the possibilities made available by division of labor. In fact, A. Smith (1976) and A. Young (1928) regarded technical progress as resulting from the division of the tasks needed to produce a certain final good into simpler ones, and the application of machinery in the execution of the latter; then the division of the tasks neede to produce machinery into simpler ones, application of machinery to these tasks, and so on *ad infinitum*. Since machinery is indivisible, a fixed cost has to be incurred at each step of the above process, so a transition to a higher stage of division of labor will take place only if output is large enough to accommodate these fixed costs. In other words, these is an infinity of intermediate goods (machinery), ranked on $[0, \infty]$ so that the furher away from zero a good v is, the more productive it is, but also the higher the fixed labor cost associated with its production. By determining the equilibrium v one knows, therefore, the equilibrium degree of technical progress, and how this varies with market size.

We can now describe the basic structure of the model: define an integrated producer (firm) as an agent who produces both the final good and the intermediate goods that are utilized in the production of the final good, and a specialist producer (firm) as an agent who produces only the final good and buys the intermediate goods that his productive activity requires. Agents play a game in three stages: given any division of agents into specialist and integrated firms, specialist producers play a game among themselves, taking supplies of integrated firms as given, and determine their demands for intermediate goods as functions of these supplies. Then, integrated firms play a game among themselves and, taking into account the demand functions established in the previous stage, determine their supplies. It turns out that all integrated firms will supply the same, unique, intermediate good v, which will depend only on the ratio m/(n-m) of specialist to integrated firms. Also, equilibrium payoffs in the second and third stage are functions of m only, the number of specialist firms. These equilibrium payoffs serve as payoff functions in the first stage of the game, which is played by all agents: the equilibrium number m of specialist firms is determined, and so, going back to the second and third stages, equilibrium demands and supplies are also determined. The economic forces that determine equilibrium values are the following: increasing returns to scale in the production of intermediate goods favors complete vertical disintegration, i.e., production of all intermediate goods by one firm, so as to minimize the average cost of producing them. On the other hand, though, strategic behavior favors complete vertical integration, i.e., production of intermediate goods by every agent, so as to minimize the losses associated with oligopolistic pricing. In other words, an integrated firm does not buy intermediate goods and so it avoids monopolistic exploitation, but on the the other hand, it has to bear alone the fixed cost of producing them; a specialist firm, on the contrary, is subject to monopolistic exploitation but shares the fixed costs of the goods it buys with the rest of specialist firms. In equilibrium, gains and losses from integration cancel out, and no agent can increase his payoff by changing his degree of integration. When the number of agents, i.e., market size, increases, the incentive to exploit scale economies by sharing fixed costs with other buyers increases, since more labor means that a lower average cost is attainable when only one firm produces intermediate goods: the ratio of specialist to integrated firms will increase, therefore, in accordance with (T1).

Consider now the degree of production roundaboutness. An integrated firm that produces a higher quality intermediate good than before increases its fixed cost and reduces its variable cost: at any given market size, there is an oprimal degree of production roundboutness, at which further increases in quality increase unit fixed cost faster than they reduce unit variable cost. The market size for each integrated firm is the proportion of demand that it serves, i.e., essentially the ratio m/(n-m) of specialist to integrated firms. When population increases, m/(n-m) increases and so the optimal degree of division of labor within firms increases, too; this agrees with (T2). Finally, notice that alternative methods of vertical control are not considered here, in the sense that agents can either integrate or not, but are not allowed to engage in any other kind of vertical relationship, like resale price maintenance, franchise fees, or volume requirements.

It has been shown in various contexts that in the absence of uncertainty and transactions costs all forms of vertical control are equivalent: Blair and Kaserman (1983, Chs. 3, 4). The present model, therefore explains the degree of vertical control rather than that of vertical integration. More structure has to be introduced in the model in order to explain which forms of vertical control, one of which is vertical integration, will prevail in equilibrium.

A. Primitives

The economy consists of n identical agents, each endowed with one unit of labor, called good o, the sole primary factor. There is only one final good, and a continuum, $R_{++} = (0, \infty)$, of intermediate goods that yield no utility but can only be used in the production of the final good. Assume that

(R1) It takes $\delta(v)$ units of inermediate good $v \in (0, \infty)$ to produce one unit of the final good q, and $\delta'(v) < 0$ on $(0, \infty)$.

(R2) It takes no labor to produce the final good.

(R3) One unit of $v \in (0, \infty)$ is produced by incurring a fixed labor cost $L(v)>0$, a variable labor cost $\ell(v) > 0$, and $a(t, v) \geq 0$ units of good $t \in (o, v)$.

By (R3) the labor cost of producing q units of good $v \in (0, \infty)$ is given by $\gamma(v)+c(v)q$ where

$$(1) \qquad c(v) = \ell(v) + \int_0^v a(t,v)\, c(t)\, dt$$

$$(2) \qquad \gamma(v) = \gamma(0) + \int_0^v L(t)\, dt.$$

We assume that (1) has a unique solution: a sufficient condition for this is $\ell, \alpha \in L_2$ (Tricomi, 1957, p. 10). Notice that (2) implies that γ is increasing, but not necessarily strictly so; we strengthen this by assuming

(R4) γ is strictly increasing on $(0, \infty)$.

Suppose that there exist $v_1 < v_2$ such that $\delta(v_1)c(v_1) \leq c(v_2)\delta(v_2)$; then, for all $q > 0$ the labor required to produce q with v is $\gamma(v)+c(v)\delta(v)q$ and so

$$\gamma(v_1) + c(v_1)\delta(v_1)q < \gamma(v_2) + c(v_2)\delta(v_2)q$$

i.e., v_2 is an inferior degree of division of labor. To ensure that the boundary of the production set is smooth, therefore, one has to exlude inferior v's, i.e., to assume

(R5) $\delta(v)c(v)$ is strictly decreasing in $v \in (0, \infty)$.

Finally, we shall assume that (c, γ) are twice continuously differentiable on $(0, \infty)$. Then, by (R4), γ^{-1} exists and so the functions

$$\psi_1 = c \circ \gamma^{-1}, \quad \psi_2 = \delta \circ \gamma^{-1}, \quad {}^*\psi = \psi_1 \circ \psi_2$$

are well defined and twice continuously differentiable on $(\gamma(0), \infty)$ and ${}^*\psi$ is strictly decreasing. The labor cost of producing q units of the final good with division of labor v is then

$$\gamma + {}^*\psi(\gamma)q, \qquad \gamma = \gamma(v)$$

By (1), $\gamma(0) = L(0)$. Assume

(R6) $\quad L(0) = 1 + \underline{\gamma} \qquad \underline{\gamma} > 0$

i.e., self-employment is impossible. Define $\psi(\gamma) = {}^*\psi(\gamma-1)$.
Then, the labor cost of producing q with γ is given by

$$L(\gamma, q) = 1 + \gamma + \psi(\gamma)q \qquad \gamma > \underline{\gamma}$$

with the obvious redefinition of γ. We will set $\psi(\gamma) = +\infty$ on $[0, \underline{\gamma}]$, i.e., on the set on which γ is not defined

(R7) $\quad \psi(\gamma) = +\infty \ \text{ on } \ [0, \gamma]$.

Further properties on ψ will be inferred by corresponding properties of the following optimization problems and their solutions:

(P1) $\quad \min\limits_{\gamma} AL(\gamma) = \dfrac{1+\gamma}{q} + \psi(\gamma), \qquad q > 0,$

i.e., the problem of minimizing the average cost of producing $q>0$ with respect to the division of labor γ. To guarantee existence and uniqueness of a continuously differentiable solution, for all $q>0$ we assume that AL is U-shaped:

$$\lim_{\gamma \to \underline{\gamma}} AL(\gamma) = +\infty,$$

and AL is strictly convex. Equivalently:

(R8) $\quad \lim\limits_{\gamma \to \underline{\gamma}} \psi(\gamma) = +\infty$

(R9) $\quad \psi''(\gamma) > 0 \ \text{ on } \ (\underline{\gamma}, \infty)$.

Consider now the problems of maximizing output q given labor supply n

$$\max q$$
$$\text{s t}\, \gamma + \psi(\gamma) q \leq n - 1 \qquad \gamma \geq 0.$$

or

(P2) $$\max_{\gamma} = \frac{n-1-\gamma}{\psi(\gamma)} \qquad \gamma \geq 0.$$

We require that a solution to (P2) exists for each $n > 1+\underline{\gamma}$. Any such solution must lie in $(\underline{\gamma}, n-1)$ because $q(\underline{\gamma}) = 0$ and $\gamma \geq n-1$ implies $q(\gamma) \leq 0$, while positive payoff is possible. Any solution, therefore, satisfies the first-order condition with equality, i.e.,

(3) $$h(\gamma) \equiv \gamma - \frac{\psi(\gamma)}{\psi'(\gamma)} = n-1$$

(R9) guarantees that (3) has a unique, strictly increasing, continuously differentiable solution $\gamma(n)$, which has to satisfy

$$\underline{\gamma} < \gamma(n) - n < 1$$

and therefore

$$\lim_{n \to \underline{\gamma}+1} \gamma(n) = \underline{\gamma}$$

which is possible if and only if, by (3),

(R10) $$\lim_{\gamma \to \underline{\gamma}} \left(-\gamma \frac{\psi'(\gamma)}{\psi(\gamma)}\right) = +\infty.$$

i.e., that the elasticity of unit variable cost reduction is very large when the degree of division of labor is small. We also want to make sure that the benefits from division of labor are not infinite when the degree of division of labor is infinite, and so we impose

(R11) $$\lim_{\gamma \to +\infty} \left(-\gamma \frac{\psi'(\gamma)}{\psi(\gamma)}\right) = \alpha < +\infty$$

and

(R12) $$\lim_{\gamma \to +\infty} \psi(\gamma) = \psi_{\infty} \geq 0.$$

Functions that satisfy these requirements include

$$\psi(\gamma) = \frac{A}{(\gamma-\bar{\gamma})^{\alpha}}\ , \qquad 0 < \alpha < +\infty\ , \qquad A > 0$$

and $$\psi(\gamma) = \psi_{\infty}\frac{\gamma}{\gamma-\bar{\gamma}}\ , \qquad \psi_{\infty} > 0\ .$$

Two other restrictions on technology will be imposed later on so as to guarantee existence of equilibria.

B. The Economic Game and Existence of Equilibrium

The economic process is modelled as a game in three stages. In the first stage, agents decide on whether to be integrated or specialist firms. In the second stage, integrated firms decide on outputs and choice of technique. In the third stage, specialist firms decide on outputs and choice of technique. At each stage, outcomes of previous stages are taken as given.

We begin at the third stage and work backwards.

Third Stage Game

This game is played among specialist firms. From the first stage a function $\beta : I \rightarrow \{1, 2\}$ is given, which indicates whether an agent $i \in I = \{1, ..., n\}$ is a specialist firm ($\beta(i)=1$) or an integrated firm ($\beta(i)=2$). The number of specialist firms is $m = m(\beta) = \#\{i \in I : \beta(i) = 1\}$.

The following magnitudes are given from the second stage: the aggregate amount of money that integrated firms spend on labor, B(o); and the aggregate supply S(v) of each good $v \in (0, \infty)$. The set $\{v \in (0, \infty) : S(v) > 0\}$ is assumed to be finite; no generality is lost, though, because integrated firms are going to supply only one good in the second stage. Specialist firms know that if B(v) is the total amount of money spent on $v \in (0, \infty)$ then $p(v)=B(v)/S(v)$, and that if S(o) is the aggregate supply of labor, then the price of labor is $p(o)=B(o)/S(o)$. Strategy spaces of specialist firms are given by

(4) $$\sum_i = \{(b(\cdot), q, s(o), \lambda(\cdot)) : q \geq 0\ , \qquad 0 \leq s(o) \leq 1\ ,$$

$$\lambda : (0, \infty) \rightarrow [0, \infty)\ , \quad b : (0, \infty) \twoheadrightarrow [0, \infty)\ ,$$

$\sum_i \lambda(v) = 1$ and $\lambda(v)$ and $b(v)$ are positive for at most

finitely many v ; $\delta(v)\,\lambda(v)\,q \leq \dfrac{b(v)}{p(v)}$.

In other words, specialist firms have to decide on how much to spend on $v \in (0, \infty)$, $b(v)$; on how much final good q to produce and how much labor $s(o)$ to supply; and on the proportion $\lambda(v)$ of the final good's output that is going to be produced by $v \in (0, \infty)$. The restriction $\delta(v)\,\lambda(v)\,q \leq b(v)/p(v)$ means that a strategy vector is feasible only if the specialist firm has obtained an amount $b(v)/p(v)$ of v which is no less than the amount $\delta(v)\,\lambda(v)\,q$ of v that goes into the production of q. Let

$$\sum_s = \Pi\{\sum_i : \beta(i) = 1, i \in I\} .$$

Then, outcome functions are given by $x : I \times [0, \infty) \times \sum \rightarrow R$

(5) $\qquad x(i, o, \sigma) = 1 - s_i(o)$

and for $v > 0$

(6) $$x(i, v, \sigma) = \begin{cases} \dfrac{b_i(v)}{p(v)} & \text{if } \sum_v b_i(v) \leq p(o) \cdot s_i(o) \\ 0 & \text{otherwise .} \end{cases}$$

In other words, when $\sigma \in \sum$ is the strategy vector played by specialist firms, firm i gets an amount of v equal to the real value of its bid for v if it is solvent, and zero otherwise.

Payoff functions $U_i : \sum \rightarrow R$ are given by

$$U_i(\sigma) = u_i\left(\sum_v \frac{x(i,v,\sigma)}{\delta(v)}\right)$$

where u_i is any strictly increasing function. In other words, labor has no disutility, and utility depends only on the amount $\sum_v \dfrac{x(i,v,\sigma)}{\delta(v)})$ of the final good obtainable under $\sigma \in \sum_S$. Without loss of generality, then, one can set all u_i equal to the identity function, since it is the argument of u_i that is maximized, and the shape of u_i itself is immaterial.

A third-stage equilibrium is a strategy vector $*\sigma \in \sum_S$ such that $\forall i \in I$ with $\beta(i)=1$ and $\forall \sigma_i \in \sum_i$.

A third stage equilibrium is symmetric if all specialist firms have identical strategy vectors. Notice that in equilibrium these is demand for all supplied goods, i.e., $S(v) > 0$ implies $B(v) > 0$, for if $B(v) = 0$, an agent can offer an infinitesimally small amount of money $b(v) > 0$ and obtain all the supply $S(v)$, since in that case he gets

$$\frac{b(v)}{p(v)} = \frac{b(v)}{B(v)} S(v) = \frac{b(v)}{b(v)} S(v) = S(v)$$

and therefore a strategy σ that involves $B(v) = 0$ and $S(v) > 0$ cannot be an equilibrium. In a symmetric equilibrium, then, each agent demands all goods, since for each good there exists at least one agent who demands it. Symmetric third stage equilibria are completely described by

Proposition 1: Let $*\sigma$ be a third-stage symmetric equilibrium.
Then $*\sigma_i = (*b(\cdot)\ *s(o), *q)$ is given by

$$(8) \qquad *b(v) = p(o) \frac{S(v) \mid \delta(v)}{\sum_u S(u) \mid \delta(u)}$$

$$(9) \qquad *s(o) = 1$$

$$(10) \qquad *q = \frac{1}{m} \sum_v \frac{S(v)}{\delta(v)}$$

Existence is proved by

Proposistion 2: Let all specialist firms but i play the strategy described by equations (8), (9) and (10). Then the optimal strategy of i is given by (8), (9) and (10).

For each pair $(S(\cdot), B(o))$, then, (8) associates the optimal level of individual demand for each v. Aggregate demand, then, is

$$(11) \qquad B(v, S, B(o)) = mp(o)\frac{S(v) \mid \delta(v)}{\sum_u S(u) \mid \delta(u)}, \qquad p(o) = \frac{B(o)}{m},$$

for each $v \in (0, \infty)$, each $B(o) \geq 0$ and each function $S : (0, \infty) \rightarrow [0, \infty)$ with at most finitely many positive values.

Second Stage Game

This is a game played among integrated firms who know the demand function (11). Each firm has to choose the amount of money b(o) it spends on labor; the amount s(v) of good v supplied; and the amount q of the final good produced. Strategy spaces are given, therefore, by

$$(12) \qquad \sum_i = \{(q, s, b(o)) : q \geq 0,\ b(o) \geq 0,\ s : (0, \infty) \rightarrow R_+$$

and s can be positive at finitely many points only;

$$\gamma(\bar{v}) + \delta(\bar{v})\, c(\bar{v})\, q + \sum_v c(v)\, s(v) \leq \frac{b(o)}{p(o)} \}.$$

where $\bar{v} = \max\{v : s(v) > 0\}$.

The inequality in the definition of $\sum_i$ states that a strategy vector is feasible ony if the amount of labor b(o)/p(o) bought by the firm is no less than the amount of labor

$$\gamma(\bar{v}) + \delta(\bar{v})\, c(\bar{v})\, q + \sum_v c(v)\, s(v)$$

consumed by the firm. Clearly, the final good will be made by the most efficient intermediate good produced, since the fixed cost $\gamma(v)$ has been incurred and $\delta(v)\, c(v)$ declines in v. Let

$$\sum_I = \Pi \{ \sum_i : \sigma(i) = 2, i \in I \}.$$

Then, outcome functions $x : \{i : \sigma(i) = 2\} \times \sum_I \rightarrow R$ are defined by

$$(6) \qquad x(i,\sigma) = \begin{cases} \dfrac{b_i(o)}{p(o)} & \text{if } \; b_i(o) \le \sum_v p(v)\, s_i(v) \\ 0 & \text{otherwise} . \end{cases}$$

In other words, a firm gets the real value of its bid for labor if it is solvent, and nothing if it is not. Outcome functions $U_i : \sum_I \rightarrow R$ are given by

$$(14) \qquad U_i(\sigma) = u_i\left[\frac{x(i,\sigma) - \gamma(\bar{v}_i) - \sum_v c(v)\, s_i(v)}{\delta(\bar{v}_i)\, c(\bar{v}_i)}\right] = u_i(q)$$

where u_i is any strictly increasing function. Again, agents do not derive utility from leisure, and so u_i can be set equal to the identity function without loss of generality. A second stage equilibrium is a strategy vector $^*\sigma \in \sum_I$ such that for any i with $\sigma(i) = 2$ and any $\sigma_i \in \sum_i$

$$(15) \qquad U_i(^*\sigma) \ge U_i(^*\sigma_1, \ldots, {}^*\sigma_{i-1}, \sigma_i, {}^*\sigma_{i+1}, \ldots, {}^*\sigma_n)$$

where $\quad \sigma_i = (b_i(o), s_i(\cdot), q_i)$.

A second-stage equilibrium is symmetric if all integrated firms have identical stategies. Symmetric second-stage equilibria are described by

Proposition 3: Let $^*\sigma$ be a symmetric second-stage equilibrium.
Then $^*\sigma_i = (^*s(\cdot), {}^*q)$ is given by

$$(16) \qquad ^*s(v) = \begin{cases} 0 & \text{if } v \neq v(m) \\ \dfrac{m(n-m-1)}{(n-m)^2 \psi_1(\gamma)} & \text{if } v = v(m) \end{cases}$$

where $\gamma = \gamma(v(m))$ and γ satisfies:

$$(17) \qquad h(\gamma) = \gamma - \frac{\psi(\gamma)}{\psi'(\gamma)} = \frac{m}{n-m}$$

$$(18) \qquad {}^*q = \frac{1}{\psi(\gamma)}\left[\frac{m}{n-m} - \gamma - \frac{m(n-m-1)}{(n-m)^2}\right].$$

Let $K(\alpha)$ be the largest integer smaller than $(1+\alpha)/\alpha$. Then

Corollary: Let K be an integer greater than $K(\alpha)$.Then there exists $N(K)$ such that $n \geq N(K)$ implies that K is not the number of integrated firms at some symmetric second-stage equilibtium.

Existence of second-stage equilibria is proved under

First Regularity Assumption: The equation system

$$h(\hat{\gamma}) = \frac{m}{n-m}$$

$$h(\gamma) = m\left(1 - \frac{n-m-1}{n-m}\left(\frac{\psi(\gamma)}{\psi(\hat{\gamma})}\right)^{1/2}\right)$$

has a unique solution $(\gamma, \hat{\gamma})$ in $(\bar{\gamma}, \infty)^2$.

Proposition 4: For each $\alpha > 0$, there exists $N = N(\alpha)$ such that $n \geq N$ implies that for each $m \in \{n - K(\alpha), ..., n-1\}$, a second-stage symmetric equilibrium exists, provided that the first regularity assumption holds.

The *Corollary* and *Proposition 4* combined show that the elasticity of cost reduction in the limit α is the crucial parameter that determines how many integrated firms can survive in equilibrium. In fact, if $\alpha \geq 1$ only one integrated firm can earn positive utility in equilibrium, however large market size is; the case of one integrated firm being both uninteresting and easy to analyze, it will be assumed from now on that $0 \leq \alpha < 1$. In fact, one would expect that the benefits from further division of labor are very small when the degree of division of labor approaches infinity, and so α should be thought of as being close to zero.

Define $\pi(1, m)$, $\pi(2, m)$ as the second-stage symmetric equilibrium payoff of specialist and integrated firms, respectively. Clearly, we can set $\pi(1, 0) = \pi(2, 0) = 0 = \pi(1, n) = \pi(2, n)$, because no production can take place either without specialist firms $(m = 0)$ or without integrated firms $(m = n)$. For those m's that a symmetric second-stage equilibrium does not exist, $\pi(i, m)$ $(i = 1, 2)$, are not defined; for the remaining m's, Propositions 1 and 3 yield

$$(19) \qquad \pi(1,m) = \frac{n-m-1}{\psi(\gamma)(n-m)}$$

$$(20) \qquad \pi(2,m) = \frac{1}{\psi(\gamma)}\left[\frac{m}{n-m} - \gamma - \frac{m(n-m-1)}{(n-m)^2}\right],$$

where γ is given by equation (17). Finally, define

$$A : \{o, ..., n-1\} \rightarrow R \text{ by}$$

$$(21) \qquad A(m) = \pi(2, m) - \pi(1, m+1)$$

The economic forces at work are the following: first, each integrated firm will offer only one intermediate good $v \in (0, \infty)$. For if a firm offers $v_1 < v_2 < ... < v_K$ it incurs a fixed cost $\gamma(v_K)$. Consider now the same firm producing only v_K: it incurs the same fixed cost, but its variable cost is lower, because $\delta(v_K)\, c(v_K) < \delta(v_i)\, c(v_i)\ \forall\ i = 1, ..., K-1$. Also, the firm, taking demand functions instead of demands as given, knows that the loss in utility due to the suspension of production of $v_1, ..., v_{K-1}$ can be made up by increasing the supply of v_K. Unit price of v_K will fall, but now the firm does not pay the high variable costs $c(v_i)\, \delta(v_i)\, s(v_i)$ but only the lowest variable cost $c(b_K)\, \delta(v_K)\, s(v_K)$. In a sense, therefore, knowledge of demand functions allows integrated firms to suspend production of inferior intermediate goods by shifting specialist firms' demand towards that good whose production is mutually advantageous. This could also be achieved if specialist firms had the strategic advantage, i.e., if they knew the supply functions of integrated firms, for then each specialist firm would adjust its demand so as to shift supply towards that good whose production is mutually advantageous. All that is needed is that one side of the game has correct information about the behaviour of the other side. Without this information, the game us identical to that of Shubik citation as described, say, by Mas-Colell (1982) which has so many equilibria as to make any predictions impossible. Secondly, the unique v produced and supplied is determined by the size of the market facing each integrated firm; by symmetry, a firm's market size is m/(n–m). If a firm increases v above its optimal level, the increase in fixed cost is not justified by the decrease in variable cost, because the size of the market is such that the additional fixed cost is not spread over enough output units to make the addition profitable. Finally, the amount of v supplied is also determined by the size of the market; the latter determines that output level at which further increases in output reduce price by more than they reduce unit cost.

The role of symmetry in the argument is twofold: first, it enormously simplifies the game because symmetric equilibria are unique and computable, and so equilibrium payoffs are functions of m only. Secondly, it allows precise measurement of the degree of vertical integration by the ratio of specialist to integrated firms, m/(n–m). By considering only symmetric equilibria we assume that when an agent changes his degree of specialization by, say, leaving the ranks of specialist firms and joining the ranks of integrated firms, two things happen: the empty space left in the ranks of specialist firms is equally divided among them; and the incumbent integrated firms make room for the new entrant (or the new entrant forces them to do so) so that all have equal market share. This seems reasonable, since there is nothing that distinguishes between incumbents and new entrants.

First Stage Game

This game is played by all agents. Strategy spaces are given by $W_i = \{1, 2\}$ and payoff functions by $H_i : W \rightarrow R$.

(22) $H_i(\beta) = \pi(\beta(i), m(\beta))$

where $W = \prod_{i=1}^{n} W_i$

and $m(\beta) = \#\{i : \beta(i) = 1\}$

In other words, an agent's payoff in the first stage is his second-stage symmetric equilibrium payoff. An equilibrium is a strategy vector $^*\beta \in W$ such that $\forall\, i \in I\ \forall\, \beta_i \in W_i$

(23) $H_i(^*\beta) \geq H_i(^*\beta_1, \ldots, {}^*\beta_{i-1}, \beta_i, {}^*\beta_{i+1}, \ldots, {}^*\beta_n)$

By equations (19), (20) and (21), $^*\beta \in W$ is an equilibrium if and only if $m = m(^*\beta)$ satisfies $A(m) \geq 0$, $A(m - 1) \leq 0$: no agent wants to change his specialization. Existence will be proved under the

Second Regularity Assumption: The roots of A are not critical points, i.e., $x \in [1, n-1]$ and $A(x) = 0$ imply $A'(x) = 0$.

Proposition 6: Let A satisfy the regularity assumption; let $0 \leq \alpha < 1$. Then there exists N that depends only on ψ such that $n \geq N$ implies that there is $m \in \{2, ..., n-2\}$ such that $A(m) \geq 0$, $A(m-1) \leq 0$, $\pi(2, n-2) > 0$, $\pi(2, 1) < 0$.

Notice that $m = n-1$ is never an equilibrium since

$$A(n-1) = \pi(2, n-1) - \pi(1, n) = \pi(2, n-1) > 0$$

$$A(n-2) = \pi(2, n-2) - \pi(1, n-1) = \pi(2, n-2) > 0 .$$

For the same reason, $m \in \{0, 1, n\}$ cannot be equilibria, i.e., there are always at least two specialist and two integrated firms.

So, the model determines the number of specialist firms, but does not specify which agents are going to be specialists; this was to be expected, since all agents are identical. The reason why an equilibrium obtains is the following: increasing returns to scale imply that unit costs would be lowest when all intermediate goods are produced by a single giant firm. Strategic behavior implies, though, that such a firm would take advantage of its monopolistic position to exploit its customers, so some of the latter will produce their own intermediate goods. In other words, increasing returns to scale tend to reduce the number of integrated firms, while strategic behavior tends to increase it. In equilibrium, the gains from producing one's own inputs (and so avoiding monopolistic exploitation) are offset by efficiency losses due to the fact that, with many integrated firms, unit cost of intermediate goods, and so their prices at any given output level is not as low as it could be.

C. Comparative Statics

The formal structure of this model and that of Vassilakis (1989) are identical, so we should expect to obtain the same formal results, with a different interpretation.

The degree of division of labor is a function of market size n in the sense of

Proposition 7: The specialist-integrated firm ratio, and the equilibrium type of intermediate good v(m) increase without bound with market size n;

$$\forall k, \exists\, N(k) : n \geq N(k) \Rightarrow \frac{m}{n-m} \geq k, \qquad v(m) \geq K .$$

Proposition 7 says that the division of labor, both within the firm and among firms, increases with market size. The reason is that with a higher population, and so a higher potential labor force, the incentive to concentrate production of intermediate goods increases, because a lower point in the average cost curve can now be attained. So, there are going to be fewer but bigger integrated firms. At the same time, since the output of each integrated firm is now larger than before, it pays to use more roundabout production methods, i.e., an intermediate good further away from zero.

We can now compare equilibria with Pareto optimal outcomes. Clearly, an allocation is Pareto optimal if and only if it maximizes total output, since there is only one good that enters utility functions. A measure of Pareto efficiency is then the ratio of actual to potential output. Actual output $\hat{Q}_n$ is given by $m\pi(1, m) + (n - m)\pi(2, m)$ for any m that satisfies $A(m) \geq 0$, $A(m-1) \leq 0$; potential output $\bar{Q}_n$ is the solution of the problem

$$\max Q$$

$$\text{s.t.}\ \gamma + \psi(\gamma)Q \leq n-1 \qquad\qquad \gamma \geq 0$$

and is given by

$$\bar{Q}_n = \frac{n-1-\bar{\gamma}_n}{\psi(\bar{\gamma}_n)}$$

where $h(\bar{\gamma}_n) = n-1.$

Proposition 8: (i) Let $0 \leq \alpha \leq 1$ and $\psi_\infty > 0$. Then the ratio of actual to potential output converges to unity as market size n increases to infinity.
(ii) Let $0 < \alpha < 1$ and $\psi_\infty = 0$. Then the ratio of actual to potential output is bounded away from unity, i.e.,

$$\lim_{n \to +\infty} \sup \frac{\hat{Q}_n}{\bar{Q}_n} < 1.$$

Proposition 9: The ratio of variable costs in equilibrium, $n - (n-m)\gamma_n$, and variable costs at the optimum, $n - \Gamma_n$, converges to unity as market size increases to infinity.

These two propositions provide a complete description of efficiency properties of equilibria in the limit. In the absence of product differentation, equilibria are inefficient in the limit if and only if $\psi_\infty = 0$ (excluding the case $\alpha = 0$, $\psi_\infty = 0$ which is inconclusive). This can be explained as follows: the ratio of actual to potential output is

$$\frac{n-(n-m)\gamma_n}{n-\Gamma_n} \cdot \frac{\psi(\Gamma_n)}{\psi(\gamma_n)}$$

i.e., the ratio of total variable labor costs divided by the ratio of unit variable cost coefficients. By proposition 9, the first ratio tends to unity, so that inefficiency, if any, is not due to misallocation between fixed and variable components of cost.

Inefficiency in the limit, therefore, will obtain if and only if the actual unit variable cost coefficient differs from the optimal one in the limit: But both actual and optimal fixed costs, namely γ_n and Γ_n, diverge to infinity, and so both actual and optimal unit variable costs converge to the same number, ψ_∞. If $\psi_\infty > 0$, then the limit of the ratio of actual to potential output is one: increases in the degree of division of labor cannot reduce variable cost below $\psi_\infty > 0$, and so returns are asymptotically constant. If, however, $\psi_\infty = 0$, then the ratio of actual to potential output is of the form 0/0 in the limit, so that the relative magnitudes of γ_n, Γ_n become important, and since γ_n is always less than Γ_n (because there are at least two integrated firms) $\psi(\gamma_n)$ is always greater than $\psi(\Gamma_n)$, even in the limit.

The proof of this last fact requires the assumption $\alpha > 0$. When $\psi_\infty = 0$ this assumption implies that returns to scale are asymptotically increasing, because both the elasticity of unit cost reduction is positive and division of labor drives unit variable cost down to zero.

Suppose now that product differentiation is allowed for: Vassilakis (1993) shows that even if $\alpha = 0$ and $\psi_\infty > 0$, inefficiency persists in the limit. The reason is that the degree of division of labor in the production of each good no longer increases to infinity with market size, because preferences for variety dictates production of more goods rather than increases in the production (and therefore in the degree of division of labor) of a given set of goods. But then actual and optimal unit variable costs are never going to be equalized, and so equilibria will be suboptimal in the limit.

D. Proofs

Proposition 1: Let $^*\sigma = (^*b(\cdot), {}^*s(o), {}^*q)$ be a third-stage symmetric strategy. For each i with $\beta(i) = 1$, $^*\sigma_i$ solves

$$\max q$$

$$\text{subject to } \delta(v)\,\lambda(v)\,q \le \frac{b(v)}{p(v)}$$

$$\textstyle\sum_v b(v) \le p(o)\cdot s(o) \qquad\qquad 0 \le s(o) \le 1$$

$$\textstyle\sum_v \lambda(v) = 1 \qquad\qquad \lambda(v) \ge 0,\, b(v) \ge 0,$$

or
$$\max q = \sum_v \frac{b(v)}{p(v)\,\delta(v)}$$

(P)
$$\text{subject to } \textstyle\sum_v b(v) \le p(o)\cdot s(o)\,, \qquad b(v) \ge 0,\ o \le s(o) \le 1\,.$$

At an equilibrium, $S(v) > 0$ implies $^*B(v) > 0$, for if $^*B(v) = 0$, then any specialist firm can obtain all of $S(v)$ by offering an arbitrarily small positive amount of money, and so $^*B(v) = 0$ cannot be an equilibrium. At a symmetric equilibrium, therefore, each specialist firm buys each and every good supplied in positive quantity, and first-order necessary conditions together with symmetry yield (7), (8) and (9).

Proposition 2: Let all specialist firms but i play the strategy (*) described by (7), (8) and (9). Then, problem (P) has a solution, because the constraint set is compact and the objective function is continuous; firm i, therefore, has an optimal response to (*), that is denoted by $b_i(\cdot)$. Let

$$U = \{u > 0 : S(u) > 0\}$$

$$V = \{v > 0 : b_i(v) > 0\}\,.$$

Clearly, $V \subset U$. Assume, for contradiction, that $V \neq U$. Then, there exists $\lambda > 0$ such that $v \in V$ implies

(A1)
$$\frac{S(v)}{\delta(v)}\,\frac{(m-1)\,^*b(v)}{[b_i(v) + (m-1)\,^*b(v)]^2} = \lambda$$

while $v \in U - V$ implies

(A2)
$$\frac{S(v)}{\delta(v)}\,\frac{1}{(m-1)\,^*b(v)} \le \lambda$$

and also

(A3) $$\sum_{v \in V} b_i(v) = p(o)\,.$$

By (A1), (A3) and (8)

(A4) $$b_i(v) = \frac{S(v)/\delta(v)}{\sum_{v \in V} S(v)/\delta(v)}\, p(o) \qquad \forall v \in V\,.$$

By (A1), (A4) and (8)

(A5) $$\lambda = \frac{m-1}{p(o) \sum_U \frac{S(u)}{\delta(u)} \left[\frac{1}{\sum_V S(v) \mid \delta(v)} + \frac{m-1}{\sum_U S(u) \mid \delta(u)}\right]^2}$$

and by (A2), (A5) and (8)

$$\frac{\sum_U \frac{S(u)}{\delta(u)}}{\sum_V \frac{S(v)}{\delta(v)}} + m - 1 \leq m - 1$$

a contradiction. Hence, $U = V$ and (A4) proves that $b_i(v) = {}^*b(v)$, i.e., the best reply of i to (*) is (*) itself.

Proposition 3: Let $^*\sigma_{-i} = ({}^*b(o), {}^*s(\cdot))$ be any second-stage strategy vector of all integrated firms except i. It is shown that i produces only one intermediate product. Suppose, for contradiction, that i produces q and $v_1 < v_2 < \dots v_K$, $K \geq 2$. Since it never pays to violate budget constraints, it is true that

(A6) $$\gamma(v_K) + c(v_K)\delta(v_K)q + \sum_{j=1}^{K} c(v_j)s(v_j) \leq \frac{b(o)}{p(o)}$$

(A7) $$b(o) = \sum_{j=1}^{K} p(v_j)s(v_j)$$

while by (8) and (9)

(A8) $$p(v_j) = \frac{B(v_j)}{S(v_j)} = \frac{mp(o)}{\delta(v_j) \sum_{v} S(u)/\delta(u)}$$

(A9) $$p(o) = \frac{B(o)}{S(o)} = \frac{B(o)}{m}\ .$$

By (A7) and (A8)

$$\frac{b(o)}{p(o)} = \frac{\sum_{i=1}^{K} s(v_j)/\delta(v_j)}{\sum_{u} S(u)/\delta(u)} m\ .$$

Hence, the integrated firm's problem is

max q

subject to $\gamma(v_K) + c(v_K)\delta(v_K)q + \sum_{j=1}^{K} c(v_j)s(v_j) \le m \dfrac{\sum_{j=1}^{K} \dfrac{s(v_j)}{\delta(v_j)}}{\sum_{u \in U} \dfrac{S(u)}{\delta(u)}}$

where $U = \{u > 0 : S(u) > 0\}$.

Consider the strategy

(A10) $$s(v_j) = \begin{cases} 0 & j = 1, \ldots, K-1 \\ s > 0 & j = K \end{cases}$$

(A11) $$q = \frac{1}{c(v_K)\delta((v_K)} \left\{ m \frac{\dfrac{s}{\delta((v_K)}}{\sum_u \dfrac{\bar{s}(u)}{\delta(u)} + \dfrac{s}{\delta((v_K)}} - \gamma((v_K) - c((v_K)s \right\}$$

where $\bar{S}(u) = \sum_{j \neq i} s_j(u)$.

To simplify notation, let

$$D = \sum_u \frac{\bar{S}(u)}{\delta(u)}$$

and

$$\text{(A12)} \qquad {}^*q = \frac{1}{c(v_K)\,S(v_K)} \left\{ m \frac{\sum_{j=1}^{K} \frac{{}^*s(v_j)}{\delta(v_j)}}{D + \sum_{j=1}^{K} \frac{{}^*s(v_j)}{\delta(v_j)}} - \gamma(v_K) - \sum_{j=1}^{K} c(v_j)\,{}^*s(v_j) \right\}.$$

It is to be shown that there exists $s > 0$ such that $q > {}^*q$. Let

$$s = \delta(v_K) \sum_{j=1}^{K} \frac{{}^*s(v_j)}{\delta(v_j)}.$$

Then

$$c(v_K)\,\delta(v_K)\,[q - {}^*q] = \sum_{j=1}^{K} c(v_j)\,{}^*s(v_j) - c(v_K)\,\delta(v_K) \sum_{j=1}^{K} \frac{{}^*s(v_j)}{\delta(v_j)} =$$

$$= \sum_{j=1}^{K} \frac{{}^*s(v_j)}{\delta(v_j)} \left[c(v_j)\,\delta(v_j) - c(v_K)\,\delta(v_K) \right] > 0$$

where the last inequality comes from

$$v_j < v_K \;\; \forall \; j = 1, \ldots, K-1$$

and the fact that δc is decreasing in v.

Hence, an integrated firm will produce at most one product, irrespectively of the strategies of other integrated firms. Assume, therefore, without loss of generality, that each integrated firm produces at most one good. First order necessary conditions for (Q) and symmetry yield (18), (16) and (17).

Corollary: Let $K > K(\alpha)$, and so $K > (1+\alpha)/\alpha$. Suppose, for contradiction, that there exists a subsequence $<n>$ of economies for which a second-stage equilibrium with K integrated firms, exists. Then, let $m = n - K$ be the number of specialist firms of these equilibria. By the definition of equilibrium $\pi(2, M) > 0$, i.e., by (20):

$$\gamma_n \leq \frac{n-k}{K} - \frac{(n-K)(k-1)}{K^2} \equiv \delta_n .$$

Equivalently, since $h' > 0$,

$$h(\gamma_n) \leq h(\delta_n)$$

and so by (16), for all n:

$$\text{(A13)} \qquad K-1 \leq \frac{\psi(\delta_n)}{\delta_n \psi'(\delta_n)} .$$

The RHS tends to $1/\alpha$, since $\delta_n \to +\infty$ as $n \to +\infty$, and so (A13) implies $K \leq (1+\alpha)/\alpha$, a contradiction.

Proposition 4: If $\alpha > 1$, then $K(\alpha) = 1$. An equilibrium with $m = n-1$ always exists, though, because there is only one integrated firm whose optimization problem is (Q). Assuming that labor supply is $m = n-1$ even if $s(v) = 0$ for all v (labor has no disutility), the integrated firm's problem, is

$$\max q$$

$$\gamma(v) + c(v)\delta(v)q \leq n-1$$

which is identical to the social optimum problem and always has a solution for $n > 1 + \bar{\gamma}$. Consider, therefore, $\alpha \in [0, 1]$ and suppose that $K \geq 2$ and that all integrated firms but i play (*), the strategy described by (15), (16) and (17). The problem of firm i is (Q), or, equivalently, to maximize

$$\text{(A14)} \qquad q = \frac{1}{\psi(\gamma)} \{ \frac{ms}{s + A\psi_2(\gamma)} - \gamma - s\psi_1(\gamma) \}$$

with respect to (s, γ), where

$$\text{(A15)} \qquad A = \frac{(n-m-1)s(v)}{\delta(v)} = m \left(\frac{n-m-1}{n-m} \right)^2 \frac{1}{\psi(\hat{\gamma})}$$

$$h(\hat{\gamma}) = \frac{m}{n-m}$$

Notice that strategy (*) yields positive utility for large n, provided that $(1+\alpha)/\alpha$ is not an integer, by the reasoning of the Corollary. If $(1+\alpha)/\alpha$ is an integer, then $K(\alpha) = (1+\alpha)/\alpha$ and so if $K = K(\alpha)$ the reasoning of the Corollary does not guarantee positive utility. Assume, therefore, that $(1+\alpha)/\alpha$ is not an integer; then, for each $K \in \{1, ..., K(\alpha)\}$ there exists $N(K)$ such that $n \geq N(K)$ implies that (*) yields positive utility. Let $N = \max\{N(K) : 1 \leq K \leq K(\alpha)\}$ and take $n \geq N$; in this way, we guarantee that i can always earn positive utility by playing (*). Notice that $\gamma > m$ yields negative utility, and so assume, without loss of generality, $\gamma \in [\bar{\gamma}, m]$: continuity of ψ_1 implies, then $\psi_1(\gamma) \geq \psi_o > 0$ and so

$$q \leq \frac{1}{\psi(\gamma)}\{m-\gamma-s\psi_o\}$$

which shows that q is negative if $s > m/\psi_o$. But q is continuous on $E = [\bar{\gamma}, m] \times [0, m/\psi_o]$ and therefore attains a maximum on E, which is also a global maximum, given that q is negative on the complement of E and that positive utility is feasible by playing (*). Any global maximum, though, has to satisfy Kuhn-Tucker conditions with strict equality, because $s = 0$ or $\gamma = 0$ yield nonpositive utility. Hence,

$$\text{(A16)} \qquad \frac{mA}{[s+A\psi_2(\gamma)]^2} = \frac{\psi_1(\gamma)}{\psi_2(\gamma)}$$

$$\text{(A17)} \qquad 1+\left[\frac{ms}{s+A\psi_2(\gamma)}-\gamma-s\psi_1(\gamma)\right]\frac{\psi'(\gamma)}{\psi(\gamma)} = -s\psi_1'(\gamma)-\frac{mAs\psi_2'(\gamma)}{[s+A\psi_2(\gamma)]^2}$$

It is shown that (A15), (A16) and (A17) have a unique solution, namely (*). By (A16)

$$\text{(A18)} \qquad s = \frac{(mA\psi_2)^{1/2}}{\psi_1} - \psi_2 A.$$

By (A17), (A18) and (A15)

$$\text{(A19)} \qquad h(\gamma) \equiv \gamma - \frac{\psi(\gamma)}{\psi'(\gamma)} = m - m\left(\frac{n-m-1}{n-m}\right)\left(\frac{\psi(\gamma)}{\psi(\hat{\gamma})}\right)^{1/2}$$

where $\hat{\gamma}$ is defined by $h(\hat{\gamma}) = \frac{m}{n-m} = \frac{n-K}{K}$.

Clearly, $\gamma = \hat{\gamma}$ solves (A19) and so (*) satisfies the first-order conditions. By the first regularity assumption $\hat{\gamma}$ is the unique solution of (A19), and so $\hat{\gamma}$ is optimal.

The rest of the proposition in this essay are proved exactly as the corresponding propositions and theorems of Vassilakis (1989), since all the information about the solutions is contained in the function A, which has the same properties in both models.

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